POINTWISE CONTROL OF THE LINEARIZED GEAR–GRIMSHAW SYSTEM

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ABSTRACT. In this paper we consider the problem of controlling pointwise, by means of a time dependent Dirac measure supported by a given point, a coupled system of two Korteweg–de Vries equations on the unit circle $T$. More precisely, by means of spectral analysis and Fourier expansion we prove, under general assumptions on the physical parameters of the system, a pointwise observability inequality which leads to the pointwise controllability when we observe two control functions. In addition, with a uniqueness property proved for the linearized system without control, we are also able to show pointwise controllability when only one control function acts internally. In both cases we can find, under some assumptions on the coefficients of the system, the sharp time of the controllability.

1. INTRODUCTION

Wave phenomena occur in many branches of mathematical physics and due to the wide practical applications it has become one of the most important scientific research areas. During the past several decades, many scientists developed mathematical models to explain the wave behavior. The Korteweg-de Vries equation (KdV),

$$u_t + u_{xxx} + uu_x = 0,$$

was first proposed as a model for propagation of unidirectional, one-dimensional, small-amplitude long waves of water in a channel. A few of the many other applications include internal gravity waves in a stratified fluid, waves in a rotating atmosphere, ion-acoustic waves in a plasma, among others. Starting in the latter half of the 1960s, the mathematical theory for such nonlinear, dispersive wave equations came to the fore as a major topic within nonlinear analysis. Since then, physicists and mathematicians were led to derive sets of equations to describe the dynamics of the waves in some specific physical regimes and much effort has been expended on various aspects of the initial
and boundary value problems. For instance, since the first coupled KdV system was proposed by Hirota and Satsuma [18, 19], it has been studied amply and some important coupled KdV models have been derived. Particularly, general coupled KdV models were applied in different fields, such as in shallow stratified liquid:

\[
\begin{aligned}
\begin{cases}
  u_t + u_{xxx} + a_3v_{xxx} + uu_x + a_1vv_x + a_2(uv)_x = 0, \\
  b_1v_t + rv_x + v_{xxx} + b_2a_3u_{xxx} + \nu v_x + b_2a_2uu_x + b_2a_1(uv)_x = 0,
\end{cases}
\end{aligned}
\]

where \( u = u(x,t) \) and \( v = v(x,t) \) are real-valued functions of the real variables \( x \) and \( t \), and \( a_1, a_2, a_3, b_1, b_2 \) and \( r \) are real constants with \( b_1 > 0 \) and \( b_2 > 0 \). System (1.1) was proposed by Gear and Grimshaw [13] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with linear coupling terms and has been object of intensive research in recent years. In particular, we refer to [3] for an extensive discussion on the physical relevance of the system in its full structure.

1.1. Setting of the problem. In this paper, we are mainly concerned with the study of the pointwise controllability of the linearized Gear–Grimshaw system posed on the unit circle \( \mathbb{T} \), namely

\[
\begin{aligned}
\begin{cases}
  u_t + u_{xxx} + av_{xxx} = f(t) & \text{in } \mathbb{R} \times \mathbb{T}, \\
  cv_t + rv_x + v_{xxx} + du_{xxx} = g(t) & \text{in } \mathbb{R} \times \mathbb{T}, \\
  u(0) = u_0, \quad v(0) = v_0, & \text{in } \mathbb{R},
\end{cases}
\end{aligned}
\]

where \( a, c, d, r \) are given positive constants and \( f, g \) are the control functions.

More precisely, the purpose is to see whether one can force the solutions of those systems to have certain desired properties by choosing appropriate control inputs. Consideration will be given to the following fundamental problem that arises in control theory, as proposed by Haraux in [15]:

**Pointwise control problem:** Given \( T > 0 \) and \((u_0, v_0), (u_T, v_T)\) in \( L^2(\mathbb{T}) \times L^2(\mathbb{T}) \), can one find appropriate \( f(t) \) and \( g(t) \) in a certain space such that the corresponding solution \((u, v)\) of (1.2) satisfies, for a fixed \( x \in \mathbb{T} \),

\[
\begin{align*}
  u(T, x) &= u_T \quad \text{and} \quad v(T, x) = v_T?
\end{align*}
\]

If one can always find control inputs to guide the system from any given initial state \((u_0, v_0)\) to any given terminal state \((u_T, v_T)\), then the system is said to be pointwise controllable.
1.2. **State of the art.** As far as we know, the internal controllability problem for system (1.1) remains open. By contrast, the study of the boundary controllability properties is considerably more developed. Indeed, the first result were obtained in [27], when the model is posed on a periodic domain and $r = 0$. In this case, a diagonalization of the main terms allows to decouple the corresponding linear system into two scalar KdV equations and use the previous results available in the literature. Later on, Micu et al. [28] proved the local exact boundary controllability property for the nonlinear system, posed on a bounded interval. Their result was improved by Cerpa and Pazoto [8] and by Capistrano–Filho et al. [6]. By considering a different set of boundary conditions, the same boundary control problem was also addressed by the authors in [7]. We note that, the results mentioned above were first obtained for the corresponding linearized systems by applying the Hilbert Uniqueness Method (HUM) due to J.-L. Lions [24], combined with some ideas introduced by Rosier in [30]. In this case, the problem is reduced to prove the so-called “observability inequality” for the corresponding adjoint system. The controllability result were then extended for the full system by means of a fixed point argument.

The internal stabilization problem has also been addressed (see, for instance, [4, 9, 29] and the references therein). Although controllability and stabilization problems are closely related, one may expect that some of the available results will have some counterparts in the context of the control problem, but this issue is open. Particularly, when the model is posed on a periodic domain, Capistrano–Filho et al. [4], designed a time-varying feedback law and established the exponential stability of the solutions in Sobolev spaces of any positive integral order by using a Lyapunov approach. This extends an earlier theorem of Dávila [9] also obtained in $H^s(T)$, for $s \leq 2$. The proofs follows the ideas introduced in [23] for the scalar KdV equation by using the infinite family of conservation laws for this equation. Such conservation lead to the construction of a suitable Lyapunov function that gives the exponential decay of the solutions. In [4], the use of the Lyapunov approach was possible thanks to the results established by Dávila and Chavez [10]. They proved that, under suitable conditions on the coefficients of the system, the system also has an infinite family of conservation laws.

1.3. **Main results.** As we mentioned before, no results about the internal controllability of the Gear-Grimshaw system are available in the literature. In this work we use spectral analysis and Fourier series to prove some results of pointwise controllability for the system (1.2).

Fourier series are considered to be very useful in linear control theory (see, e.g. [31] and its references). In particular, a classical generalization of Parseval’s equality, given by Ingham [20], and its many recent
variants are very efficient in solving many control problems where other methods do not seem to apply. An outline of this theory is presented in [1, 22, 25].

Here, we also prove some new results concerning the use of harmonic analysis in the framework of dispersive systems. In this spirit, we derive the controllability of the linearized Gear-Grimshaw system posed on the unit circle $\mathbb{T}$. On the other hand, it was pointed out earlier by Haraux and Jaffard [15, 16, 17] that controllability properties depend heavily on the location of the observation or control point.

Thus, in this work, we study the Gear–Grimshaw system with pointwise control,

\begin{equation}
\begin{cases}
  u_t + u_{xxx} + av_{xxx} = f(t)\delta_{x_0} & \text{in } \mathbb{R} \times \mathbb{T}, \\
  v_t + \frac{c}{\epsilon} v_x + \frac{1}{\epsilon} v_{xxx} + \frac{d}{\epsilon} u_{xxx} = g(t)\delta_{x_0} & \text{in } \mathbb{R} \times \mathbb{T}, \\
  u(0) = u_0, & v(0) = v_0,
\end{cases}
\end{equation}

where $a, c, d, r$ are given positive constants, $\delta_{x_0}$ denotes the Dirac delta function centered in a given point $x_0 \in \mathbb{T}$, and $f, g$ are the control functions.

One of the main result provides a sharp positive answer for the controllability issue mentioned in the beginning of this introduction.

**Theorem 1.1.** For almost all quadruples $(a, c, d, r) \in (0, \infty)^4$ the following property holds. For any fixed $x_0 \in \mathbb{T}$, $T > 0$ and $(u_0, v_0), (u_T, v_T) \in H := L^2(\mathbb{T}) \times L^2(\mathbb{T})$ there exist control functions $f, g \in L^2_{loc}(\mathbb{R})$ such that the solution of (1.3) satisfies the final conditions

$$u(T) = u_T \quad \text{and} \quad v(T) = v_T.$$

We prove this theorem by applying the Hilbert Uniqueness Method (HUM) due to J.-L. Lions [24] (see also Dolecki and Russell [12]) that reduces the controllability property to the observability of the homogeneous dual problem

\begin{equation}
\begin{cases}
  u_t + u_{xxx} + av_{xxx} = 0 & \text{in } \mathbb{R} \times \mathbb{T}, \\
  v_t + \frac{c}{\epsilon} v_x + \frac{1}{\epsilon} v_{xxx} + \frac{d}{\epsilon} u_{xxx} = 0 & \text{in } \mathbb{R} \times \mathbb{T}, \\
  u(0) = u_0, & v(0) = v_0.
\end{cases}
\end{equation}

More precisely, Theorem 1.1 will be obtained as a consequence of

**Theorem 1.2.** For almost all quadruples $(a, c, d, r) \in (0, \infty)^4$ the following properties hold.

(i) Given any $(u_0, v_0) \in H$, the system (1.4) has a unique solution $(u, v) \in C_b(\mathbb{R}, H)$, and the linear map

$$(u_0, v_0) \mapsto (u, v)$$

is continuous from $H$ into $C_b(\mathbb{R}, H)$. 

(ii) The energy of the solution, defined by the formula

$$E(t) := \|(u(t), v(t))\|_H^2 = \int_T |u(t,x)|^2 + \frac{ac}{d} |v(t,x)|^2 \, dx,$$

does not depend on $t \in \mathbb{R}$.

(iii) For every solution and $x_0 \in \mathbb{T}$ the functions $u(\cdot, x_0)$ and $v(\cdot, x_0)$ are well defined in $L^2_{\text{loc}}(\mathbb{R})$.

(iv) For every non-degenerate bounded interval $I$ there exist two positive constants $\alpha, \beta$ such that

$$\alpha E \leq \int_I |u(t,x_0)|^2 + |v(t,x_0)|^2 \, dt \leq \beta E$$

for all solutions of (1.4) and for all $x_0 \in \mathbb{T}$.

By applying a general method [21], analogous to HUM, Theorem 1.2 will also imply the pointwise exponential stabilizability of (1.3):

**Theorem 1.3.** For almost all quadruples $(a,c,d,r) \in (0, \infty)^4$ the following property holds. For any fixed $x_0 \in \mathbb{T}$, $T > 0$ and $\omega > 0$ there exist two continuous linear maps

$$F : H \to L^2(0,T) \quad \text{and} \quad G : H \to L^2(0,T)$$

such that, extending $F(u,v)$ and $G(u,v)$ by zero outside $(0,T)$, the following properties hold.

(i) Given any $(u_0, v_0) \in H$, the system

$$\begin{cases}
  u_t + u_{xxx} + av_{xxx} = F(u,v)\delta_{x_0} & \text{in} \quad \mathbb{R} \times \mathbb{T}, \\
  v_t + \frac{c}{e}v_x + \frac{1}{c}v_{xxx} + \frac{d}{c}u_{xxx} = G(u,v)\delta_{x_0} & \text{in} \quad \mathbb{R} \times \mathbb{T}, \\
  u(0) = u_0, \quad v(0) = v_0
\end{cases}$$

has a unique solution $(u,v) \in C_b(\mathbb{R}, H)$, and the linear map $(u_0, v_0) \mapsto (u,v)$ is continuous from $H$ into $C_b(\mathbb{R}, H)$.

(ii) There exists a constant $M > 0$ such that

$$\|(u(t), v(t))\|_H \leq Me^{-\omega t} \|(u_0, v_0)\|_H$$

for all solutions and for all $t \geq 0$.

Another relevant result of this work is a uniqueness result when only one function, $u(\cdot, x_0)$ or $v(\cdot, x_0)$, is observed.

**Theorem 1.4.** For almost all quadruples $(a,c,d,r) \in (0, \infty)^4$ the following property holds.

Fix $x_0 \in \mathbb{T}$ and a non-degenerate interval $I$ arbitrarily, and consider a solution of (1.4).

(i) If $u(t,x_0) = 0$ for all $t \in I$, then $u = 0$ and $v$ is an arbitrary constant function.
(ii) If $v(t, x_0) = 0$ for all $t \in I$, then $v = 0$ and $u$ is an arbitrary constant function.

Remarks.

(i) Theorem 1.1, 1.2, 1.3 and 1.4 will be proved in sharper forms, by making the assumptions of $a, c, d, r$ more explicit, and considering also some cases where the results hold only under a sharp condition $|I| > T_0$ or $T > T_0$ with some explicitly given $T_0$, where $|I|$ denotes the length of the interval $I$.

(ii) Theorem 1.4 will be obtained as a corollary to a weakened observability result. The latter implies, similarly to Theorem 1.2, some weakened exact controllability and exponential stabilizability results by acting only in one of the equations.

(iii) The above results remains valid for the scalar KdV equation. We will not present the proofs because they are similar (and simpler) than the proofs given in this paper.

(iv) If we require more regularity on the initial data, say $(u_0, v_0) \in H^3(T) \times H^3(T)$, then the results obtained for the linear system allow us to prove the local controllability of the nonlinear system by means of a fixed point argument. The proof is similar to that of [5, Theorem 2.2], and hence it will be omitted.

The plan of the present article is the following. In Section 2 we present some known and new vectorial Ingham type theorems which will form the basis of the proofs of our observability and uniqueness theorems.

Then Theorems 1.2, 1.1, 1.3 and 1.4 will be proved (in strengthened forms) in Sections 3, 4, 5 and 6, respectively.

2. SOME VECTORIAL INGHAM TYPE THEOREMS

First we recall a classical theorem of Ingham [20]. Given a family $(\omega_k)_{k \in K}$ of real numbers, we consider functions of the form

\[ \sum_{k \in K} c_k e^{i \omega_k t} \]

with square summable complex coefficients $c_k$, and we investigate the relationship between the quantities

\[ \int_I \left| \sum_{k \in K} c_k e^{i \omega_k t} \right|^2 dt \quad \text{and} \quad \sum_{k \in K} |c_k|^2. \]

Following Vinogradov, here and in the sequel we use the notation $A \ll B$ for two quantities $A$ and $B$ if there exists a positive constant $\alpha$ satisfying $A \leq \alpha B$ for all square summable families $(c_k)_{k \in K}$ of complex numbers, and we write $A \asymp B$ if $A \ll B$ and $B \ll A$. 
Theorem 2.1 (Ingham). Assume that the family \((\omega_k)_{k \in K}\) is uniformly separated, i.e.,
\[
\gamma := \inf \left\{ |\omega_k - \omega_n| : k \neq n \right\} > 0.
\]
(i) The sum (2.1) is a well defined function in \(L^2_{\text{loc}}(\mathbb{R})\) for every square summable family \((c_k)_{k \in K}\) of complex numbers.
(ii) We have
\[
\int_I \left| \sum_{k \in K} c_k e^{i\omega_k t} \right|^2 dt \ll \sum_{k \in K} |c_k|^2
\]
for every bounded interval \(I\).
(iii) We have
\[
\sum_{k \in K} |c_k|^2 \ll \int_I \left| \sum_{k \in K} c_k e^{i\omega_k t} \right|^2 dt
\]
for every bounded interval \(I\) of length \(> \frac{2\pi}{\gamma}\).

The inequalities in (ii) and (iii) are called direct and inverse inequalities, respectively.

Next we recall from [1] a generalization of Ingham’s theorem. Let \((\omega_k)_{k \in \mathbb{Z}}\) be an increasing sequence, satisfying for some \(M \geq 1\) the weakened gap condition
\[
(2.2) \quad \gamma_M := \inf_k \frac{\omega_{k+M} - \omega_k}{M} > 0.
\]
(For \(M = 1\) this is the uniform gap condition of Theorem 2.1.)

Fix \(0 < \varepsilon \leq \gamma_M\) arbitrarily. For each maximal chain \(\omega_k, \ldots, \omega_n\) satisfying
\[
\omega_{j+1} - \omega_j < \varepsilon \quad \text{for} \quad j = k + 1, \ldots, n
\]
(a chain has at most \(M\) elements), we introduce the divided differences
\[
e_k(t) := e^{i\omega_k t}, \quad e_{k+1}(t) := e^{i\omega_{k+1} t} - e^{i\omega_k t}, \ldots, e_n(t) := \ldots,
\]
and we rewrite the usual exponential sums in the form
\[
\sum_{k \in \mathbb{Z}} c_k e^{i\omega_k t} = \sum_{k \in \mathbb{Z}} b_k c_k(t).
\]
Furthermore, we set
\[
\gamma_\infty := \sup_{N = 1, 2, \ldots} \inf_k \frac{\omega_{k+N} - \omega_k}{N}.
\]
Note that \(\gamma_\infty \geq \gamma\) with \(\gamma\) defined as in Theorem 2.1. It may be shown that
\[
\gamma_\infty = \lim_{N \to \infty} \inf_k \frac{\omega_{k+N} - \omega_k}{N}.
\]
Remark. Sometimes it is easier to compute the critical length by using the upper density \( D^+ = D^+(\Omega) \) of the family \( \Omega := \{ \omega_k \} \), defined as follows.

For each \( \ell > 0 \) we denote by \( n^+(\ell) \) the largest number of exponents \( \omega_k \) that we may find in an interval of length \( \ell \), and then we set

\[
D^+ := \inf_{\ell > 0} \frac{n^+(\ell)}{\ell}.
\]

It can be shown that

\[
D^+ := \lim_{\ell \to \infty} \frac{n^+(\ell)}{\ell} \quad \text{and} \quad \frac{2\pi}{\gamma_\infty} = 2\pi D^+;
\]

See, e.g., [1, p. 57 and Proposition 1.4, p. 59] or [22, p. 174 and Proposition 9.3, p. 175].

It follows easily from the definition that \( D^+ \) is subadditive, i.e,

\[
D^+(\Omega_1 \cup \Omega_2) \leq D^+(\Omega_1) + D^+(\Omega_2)
\]

for any families \( \Omega_1 \) and \( \Omega_2 \).

Theorem 2.2. Assume (2.2), and use the above notations.

(i) The sum (2.1) is a well defined function in \( L^2_{\text{loc}}(\mathbb{R}) \) for every square summable family \( (c_k)_{k \in K} \) of complex numbers.

(ii) We have

\[
\int_I \left| \sum_{k \in \mathbb{Z}} a_k e^{i \omega_k t} \right|^2 dt \ll \sum_{k \in \mathbb{Z}} |b_k|^2
\]

for all bounded intervals \( I \).

(iii) We have

\[
\sum_{k \in \mathbb{Z}} |b_k|^2 \ll \int_I \left| \sum_{k \in \mathbb{Z}} a_k e^{i \omega_k t} \right|^2 dt
\]

for all bounded intervals \( I \) of length \( > \frac{2\pi}{\gamma_\infty} = 2\pi D^+ \).

Remarks.

- Mehrenberger [26] proved that \( \frac{2\pi}{\gamma_\infty} = 2\pi D^+ \) is the critical length for the validity of the inverse inequality.
- If the sequence \( (\omega_k) \) has a uniform gap, i.e., \( \gamma > 0 \), then choosing \( \varepsilon \leq \gamma \) every chain is a singleton, so that \( b_k = a_k \) for all \( k \). In this special case Theorem 2.2 reduces to an earlier theorem of Beurling [2].

We will also need some vectorial extensions of Theorem 2.1. Let us consider a finite number of families of real numbers:

\[
(\omega_{1,k})_{k \in K_1}, \ldots, (\omega_{J,k})_{k \in K_J}
\]

and corresponding vectors \( Z_{j,k} \) in some Hilbert space \( H \).
Theorem 2.3. Assume that each family in (2.3) has a uniform gap, i.e.,

\[ \gamma_j := \inf \{ \omega_{j,k} - \omega_{j,n} : k \neq n \} > 0 \quad \text{for} \quad j = 1, \ldots, J. \]

Furthermore assume that the family of vectors \( Z_{j,k} \) is uniformly bounded, i.e.,

\[ M_j := \sup_k \| Z_{j,k} \| < \infty, \quad j = 1, \ldots, J. \]

(i) The sum

\[ \sum_{j=1}^{J} \sum_{k \in K_j} c_{j,k} e^{i\omega_{j,k} Z_{j,k}} \]

is a well defined element of \( L^2_{\text{loc}}(\mathbb{R}, H) \) for every square summable family \( (c_k)_{k \in K} \) of complex numbers.

(ii) We have

\[ \int_I \left| \sum_{j=1}^{J} \sum_{k \in K_j} c_{j,k} e^{i\omega_{j,k} Z_{j,k}} \right|^2 dt \ll \sum_{j=1}^{J} M_j^2 \sum_{k \in K_j} |c_{j,k}|^2 \]

for every bounded interval \( I \).

Proof. We may easily adapt the usual proof of Theorem 2.1 for the special case where \( Z_{j,k} = 1 \) for all \( j, k \). See, e.g., the proof of [22, Theorem 6.1, p. 90]. \( \square \)

Under some extra assumptions the inverse inequality also holds:

Theorem 2.4. Assume the hypotheses of the preceding proposition. Moreover, assume that there exist linearly independent vectors \( Z_1, \ldots, Z_J \) satisfying the conditions

\[ \delta := \max_j \inf_{k \in K_j} \| Z_{j,k} - Z_{j} \| < \infty. \]

Then for every bounded interval \( I \) of length \( |I| > \max_j \frac{2\pi}{\gamma_j} \) there exist two positive constants \( \alpha, \beta \), independent of \( \delta \), such that

\[ \int_I \left| \sum_{j=1}^{J} \sum_{k \in K_j} c_{j,k} e^{i\omega_{j,k} Z_{j,k}} \right|^2 dt \geq (\beta - \alpha \delta^2) \sum_{j=1}^{J} \sum_{k \in K_j} |c_{j,k}|^2 \]

for all square summable sequences

\( (c_{1,k})_{k \in K_1}, \ldots, (c_{J,k})_{k \in K_J} \)

of complex numbers.

A similar theorem has been proved by Delage [11].
Proof. Using Young’s inequality we have

\begin{equation}
\left(2.4\right) \int_I \left| \sum_{j=1}^J \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} Z_{j,k} \right|^2 dt \geq \frac{1}{2} \int_I \left| \sum_{j=1}^J \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} Z_{j} \right|^2 dt
- \int_I \left| \sum_{j=1}^J \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} (Z_{j} - Z_{j,k}) \right|^2 dt.
\end{equation}

Since the vectors $Z_j$ are linearly independent, there exists a positive constant $\beta_1$ such that

\[ \frac{1}{2} \left| \sum_{j=1}^J \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} Z_{j} \right|^2 \geq \beta_1 \left| \sum_{j=1}^J \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} \right|^2. \]

Using this inequality and applying the preceding proposition to the last integral in (2.4) we obtain with some positive constant $\alpha$ that

\[ \int_I \left| \sum_{j=1}^J \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} Z_{j,k} \right|^2 dt \geq \beta_1 \left| \sum_{j=1}^J \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} \right|^2
- \alpha \delta^2 \sum_{j=1}^J \sum_{k \in K_j} |c_{j,k}|^2. \]

Since $|I| > \max_j \frac{2\pi}{\gamma_j}$, by Theorem 2.1 there exist positive constants $\varepsilon_1, \ldots, \varepsilon_J$ such that

\[ \int_I \left| \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} \right|^2 dt \geq \varepsilon_j \sum_{k \in K_j} |c_{j,k}|^2, \quad j = 1, \ldots, J, \]

and therefore

\[ \int_I \left| \sum_{j=1}^J \sum_{k \in K_j} c_{j,k} e^{i \omega_{j,k} t} Z_{j,k} \right|^2 dt \geq \beta_1 \left( \sum_{j=1}^J \varepsilon_j \sum_{k \in K_j} |c_{j,k}|^2 - \alpha \delta^2 \sum_{j=1}^J \sum_{k \in K_j} |c_{j,k}|^2 \right)
= \sum_{j=1}^J (\beta_1 \varepsilon_j - \alpha \delta^2) \sum_{k \in K_j} |c_{j,k}|^2. \]

Hence the proposition follows with

$\beta := \beta_1 \min \{ \varepsilon_1, \ldots, \varepsilon_J \}$. \hfill \Box

Remark. Applying Theorem 2.2 instead of Theorem 2.1 the proposition remains valid under the weaker assumption $|I| > 2\pi \max \left\{ D^+_{\gamma_1}, \ldots, D^+_{\gamma_J} \right\}$ instead of $|I| > \max_j \frac{2\pi}{\gamma_j}$, where $D^+_{\gamma_j}$ denotes the upper density of the family $(\omega_{j,k})_{k \in K_j}$ for $j = 1, \ldots, J$. 
Next we recall a vectorial generalization of a powerful estimation of Haraux [14]. Let we are given the families of exponents (2.3) and corresponding vectors $Z_{j,k}$ in a Hilbert space $H$. The following proposition is a special case of [22, Theorem 6.2]:

**Theorem 2.5.** Assume that there exist finite subsets $F_j \subset K_j$ and a bounded interval $I_0$ such that

$$
\int_{I_0} \left| \sum_{j=1}^{J} \sum_{k \in K_j \setminus F_j} c_{j,k} e^{i\omega_{j,k}} Z_{j,k} \right|^2 dt \asymp \sum_{j=1}^{J} \sum_{k \in K_j \setminus F_j} |c_{j,k}|^2
$$

for all square summable families

$$
(\omega_{1,k})_{k \in K_1 \setminus F_1}, \ldots, (\omega_{J,k})_{k \in K_J \setminus F_J}.
$$

Furthermore, assume that for each exceptional index $(j,k)$ the vector $Z_{j,k}$ is non-zero, and the exponent $\omega_{j,k}$ has a positive distance from the sets

$$
\{ \omega_{j,n} : n \in K_j \setminus \{k\} \}
$$

and

$$
\{ \omega_{\ell,n} : n \in K_{\ell} \} \text{ for all } \ell \neq j.
$$

Then for each bounded interval $I$ of length $|I_0|$ the relation

$$
\int_{I_0} \left| \sum_{j=1}^{J} \sum_{k \in K_j} c_{j,k} e^{i\omega_{j,k}} Z_{j,k} \right|^2 dt \asymp \sum_{j=1}^{J} \sum_{k \in K_j} |c_{j,k}|^2
$$

holds for all square summable families (2.3).

3. **Pointwise observability**

Given four positive constants $a, c, d, r$, we consider the following system of linear partial differential equations with $2\pi$-periodic boundary conditions:

$$
\begin{align*}
\frac{\partial u}{\partial t} + u_{xxx} + av_{xxx} &= 0 \quad \text{in} \quad \mathbb{R} \times (0, 2\pi), \\
\frac{\partial v}{\partial t} + rv_x + v_{xxx} + du_{xxx} &= 0 \quad \text{in} \quad \mathbb{R} \times (0, 2\pi), \\
\frac{\partial^2 u}{\partial t^2} (t, 0) &= \frac{\partial^2 u}{\partial t^2} (t, 2\pi) \quad \text{for} \quad t \in \mathbb{R}, \quad j = 0, 1, 2, \\
\frac{\partial^2 v}{\partial t^2} (t, 0) &= \frac{\partial^2 v}{\partial t^2} (t, 2\pi) \quad \text{for} \quad t \in \mathbb{R}, \quad j = 0, 1, 2, \\
u(0, x) &= u_0(x) \quad \text{for} \quad x \in (0, 2\pi), \\
v(0, x) &= v_0(x) \quad \text{for} \quad x \in (0, 2\pi).
\end{align*}
$$

It will be more convenient to write $u(t)(x) := u(t, x)$, and to work on the unit circle $\mathbb{T}$ without boundary conditions, i.e., to rewrite our system in the form

POINTWISE CONTROL
\begin{align}
\begin{cases}
u_t + u_{xxx} + av_{xxx} = 0 \quad \text{in} \quad \mathbb{R} \times T, \\
cv_t + rv_x + v_{xxx} + du_{xxx} = 0 \quad \text{in} \quad \mathbb{R} \times T, \\
u(0) = u_0 \quad \text{and} \quad v(0) = v_0.
\end{cases}
\end{align}

Writing $'$ for the time derivative and $D$ instead of the spatial derivative $\partial_x$, we may also write (3.1) in the abstract form

$$Z' + AZ = 0, \quad Z(0) = Z_0$$

in the Hilbert space

$$H := L^2(T) \times L^2(T)$$

with

$$Z = \begin{pmatrix} u \\ v \end{pmatrix}, \quad Z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

and the linear operator

$$A = \frac{1}{c} \begin{pmatrix} cD^3 & acD^3 \\ dD^3 & rdD + D^3 \end{pmatrix}, \quad D(A) := H^3(T) \times H^3(T).$$

The well posedness of (3.1) in most cases follows from the

Proposition 3.1. If $ad \neq 1$, then $A$ is an anti-adjoint operator in $H$ for the Euclidean norm given by

$$\|(z_1, z_2)\|^2 := \int_T |z_1|^2 + \frac{ac}{d} |z_2|^2 \, dx,$$

and hence it generates a group of isometries in $H$.

Proof. It is clear that $D(A)$ is dense in $H$. We have to prove that $A^* = -A$. First we show that $-A \subset A^*$, i.e., $(u, v) \in D(A^*)$ and $A^*(u, v) = -A(u, v)$ for all $(u, v) \in D(A))$. Indeed, for any $(\varphi, \psi), (u, v) \in D(A)$, we obtain by integrating by parts the equality

$$((\varphi, \psi), A(u, v))_H = -\int_T u(\varphi_x + a\psi_{xxx}) \, dx \\
- \frac{a}{d} \int_T \varphi_xv_x + \psi_{xxx} + d\varphi_{xxx} \, dx = -(A(\varphi, \psi), (u, v))_H.$$

It remains to show that $D(A^*) = D(-A)$, i.e., that each $(\varphi, \psi) \in D(A^*)$ belongs to $D(A)$.

Pick any $(\varphi, \psi) \in D(A^*)$. Then there exists a constant $C > 0$ satisfying for all $(u, v) \in D(A)$ the inequality

$$|((\varphi, \psi), A(u, v))_H| \leq C \|(u, v)\|_H,$$
or equivalently
\[
\left| \int_\mathbb{T} \varphi(u_{xxx} + av_{xxx}) + \frac{a}{d}\psi(rv_x + v_{xxx} + du_{xxx})dx \right| \\
\leq C \left( \int_\mathbb{T} \left[ u^2 + \frac{ac}{d}v^2 \right] dx \right)^{\frac{1}{2}}.
\]

Choosing \( v = 0 \) and \( u \in C^\infty(\mathbb{T}) \) hence we infer from that the distributional derivative \( \varphi_{xxx} + a\psi_{xxx} \) belongs to \( L^2(\mathbb{T}) \). Similarly, choosing \( u = 0 \) and \( v \in C^\infty(\mathbb{T}) \) we obtain that the distributional derivative \( d\varphi_{xxx} + rv_x + \psi_{xxx} \) belongs to \( L^2(\mathbb{T}) \). Combining the two relations we obtain that \((1 - ad)\psi_{xxx} + r\psi_x \in L^2(\mathbb{T})\), and in case \( ad \neq 1 \) this implies that \( \psi \in H^3(\mathbb{T}) \). Combining this property with the relation \( \varphi_{xxx} + a\psi_{xxx} \in L^2(\mathbb{T}) \) we conclude that \( \varphi_{xxx} \in L^2(\mathbb{T}) \), and therefore \( \varphi \in H^3(\mathbb{T}) \).

□

In order to establish the well posedness of (3.1) in the general case we determine the eigenvalues and eigenfunctions of the operator \( A \).

There exists an orthogonal basis
\begin{equation}
(3.2) \quad e^{ikx}Z_k^\pm, \quad k \in \mathbb{Z}
\end{equation}
of \( H \), consisting of eigenfunctions of \( A \). Indeed, for each fixed \( k \), \( e^{ikx}(u_k, v_k) \) is an eigenvector of \( A \) with the eigenvalue \( i\omega_k \) if and only if
\[
\begin{cases}
(\omega_k - k^3)u_k - ak^3v_k = 0, \\
-dk^3u_k + (c\omega_k + rk - k^3)v_k = 0.
\end{cases}
\]

There exist non-trivial solutions if and only if
\[
\begin{vmatrix}
\omega_k - k^3 & -ak^3 \\
-dk^3/c & \omega_k + rk/c - k^3/c
\end{vmatrix} = 0,
\]
or equivalently if
\[
c\omega_k^2 + (rk - (c + 1)k^3)\omega_k + (1 - ad)k^6 - rk^4 = 0.
\]

Hence we have two possible exponents, given by the formula
\begin{equation}
(3.3) \quad 2c\omega_k^\pm = (c + 1)k^3 - rk \pm k\sqrt{4acdk^4 + [(c - 1)k^2 + r]^2}.
\end{equation}

If \( k \neq 0 \), then \( \omega_k^- \neq \omega_k^+ \), and two corresponding non-zero eigenvectors are given by the formula
\begin{equation}
(3.4) \quad Z_k^\pm := 2ck^{-3}(ak^3, \omega_k^\pm - k^3)
\end{equation}

\[
= \left( 2ac, 1 - c - r k^{-2} \pm \sqrt{4ac + [c - 1 + rk^{-2}]^2} \right).
\]

If \( k = 0 \), then both eigenvalues are equal to zero, and two linearly independent eigenvectors are given for example by the formula
\begin{equation}
(3.5) \quad Z_0^\pm := \left( 2ac, \pm\sqrt{4ac} \right).
\end{equation}
Lemma 3.2. The functions (3.2) given by (3.3)–(3.5) form an orthogonal basis in $H$, and
\[ AZ_k^\pm = i\omega_k^\pm Z_k^\pm \]
for all $k \in \mathbb{Z}$.

Furthermore, we have
\[ Z_k^\pm \to Z^\pm := \left(2ac, 1 - c \pm \sqrt{4acd + (c - 1)^2}\right) \quad \text{as} \quad k \to \pm \infty, \]
and $\|Z_k^\pm\| \asymp 1$.

Proof. The orthogonality follows from the orthogonality of the functions $e^{ikx}$ in $L^2(\mathbb{T})$ and from the orthogonality relations
\[ Z_k^+ \cdot Z_k^- = 0 \quad \text{for all} \quad k \in \mathbb{Z}. \]
The latter equalities may be checked directly: we have
\[ Z_0^+ \cdot Z_0^- = 4c^2a^2 + \frac{ac}{d}(-4acd) = 0, \]
and
\[ Z_k^+ \cdot Z_k^- = 4c^2a^2 + \frac{ac}{d} \left[\left[1 - c - rk^{-2}\right]^2 - 4acd - \left[c - 1 + rk^{-2}\right]^2\right] = 0 \]
if $k \neq 0$.

The limit relations readily follow from (3.4) because $k^{-2} \to 0$. Since $Z^\pm \neq 0$, they imply the property $\|Z_k^\pm\| \asymp 1$. \qed

Next we establish the well posedness of (3.1). Let us denote by $C_b(\mathbb{R}, H)$ the Banach space of bounded continuous functions $\mathbb{R} \to H$ for the uniform norm.

Theorem 3.3.

(i) Given any $(u_0, v_0) \in H$, the system (3.1) has a unique solution $(u, v) \in C_b(\mathbb{R}, H)$, and the linear map
\[ (u_0, v_0) \mapsto (u, v) \]
is continuous from $H$ into $C_b(\mathbb{R}, H)$.

(ii) The energy of the solution, defined by the formula
\[ E(t) := \|(u, v)(t)\|^2_H = \int_T |u(t, x)|^2 + \frac{ac}{d} |v(t, x)|^2 \ dx, \]
does not depend on $t \in \mathbb{R}$.

(iii) The solution is given by the explicit formula
\[ \left( \begin{array}{c} u \\ v \end{array} \right) (t, x) = \sum_{k \in \mathbb{Z}} \left( c_k^+ e^{i\omega_k^+ t} Z_k^+ + c_k^- e^{i\omega_k^- t} Z_k^- \right) e^{ikx} \]
with suitable square summable complex coefficients $c_k^\pm$ satisfying the equality
\[ \sum_{k \in \mathbb{Z}} \left( |c_k^+|^2 \cdot \|Z_k^+\|^2 + |c_k^-|^2 \cdot \|Z_k^-\|^2 \right) = 2\pi E, \]
and hence the relation

\[(3.8) \quad \sum_{k \in \mathbb{Z}} \left( |c_k^+|^2 + |c_k^-|^2 \right) \asymp E.\]

Proof. If \(ad \neq 1\), then the theorem follows from Proposition 3.1 and Lemma 3.2.

The following alternative proof works even if \(ad = 1\). The functions \((2\pi)^{-1} e^{ikx}, k \in \mathbb{Z}\) form an orthonormal basis in \(L^2(\mathbb{T})\). Furthermore, by Lemma 3.2 the non-zero vectors \(Z_k^\pm\) are orthogonal in \(C^2\), and \(\|Z_k^\pm\| \asymp 1\). Hence the functions \(e^\pm_k(x) := e^{ikx}Z_k^\pm, k \in \mathbb{Z}\) form an orthogonal basis in \(H\), and \(\|e_k^\pm\| \asymp 1\).

By a standard method, the theorem will be proved if we show that for any given square summable sequences \((c_k^\pm)\) the series

\[U(t) := \sum_{k \in \mathbb{Z}} \left( c_k^+ e^{i\omega_k^+ t} Z_k^+ + c_k^- e^{i\omega_k^- t} Z_k^- \right) e^{ikx}\]

is uniformly convergent in \(C_b(\mathbb{R}, H)\), and \(\|U(t)\|\) is independent of \(t\). Since \(C_b(\mathbb{R}, H)\) is a Banach space, it suffices to check the uniform Cauchy criterium. Setting

\[U_n(t) := \sum_{k=-n}^{n} \left( c_k^+ e^{i\omega_k^+ t} Z_k^+ + c_k^- e^{i\omega_k^- t} Z_k^- \right) e^{ikx}, \quad n = 1, 2, \ldots,\]

the following equality holds for all \(n > m > 0\) and \(t \in \mathbb{R}\):

\[U_n(t) - U_m(t) = \sum_{m < |k| \leq n} \left( c_k^+ e^{i\omega_k^+ t} Z_k^+ + c_k^- e^{i\omega_k^- t} Z_k^- \right) e^{ikx}.\]

Using the above mentioned orthogonality properties and the relation \(\|Z_k^\pm\| \asymp 1\) hence we infer that

\[
\|U_n(t) - U_m(t)\|_H^2 = 2\pi \sum_{m < |k| \leq n} \left( \|c_k^+ e^{i\omega_k^+ t} Z_k^+\|_{C^2}^2 + \|c_k^- e^{i\omega_k^- t} Z_k^-\|_{C^2}^2 \right)
\]

\[= 2\pi \sum_{m < |k| \leq n} \left( \|c_k^+ Z_k^+\|_{C^2}^2 + \|c_k^- Z_k^-\|_{C^2}^2 \right)
\]

\[\asymp \sum_{m < |k| \leq n} \left( |c_k^+|^2 + |c_k^-|^2 \right).\]

The Cauchy property follows by observing that the last expression is independent of \(t\), and that it converges to zero as \(n > m \to \infty\) by the convergence of the numerical series

\[\sum_{k \in \mathbb{Z}} \left( |c_k^+|^2 + |c_k^-|^2 \right).
\]

The above proof also yields (by taking \(m = -1\)) the equalities (3.6)–(3.8). \(\square\)
Now we turn to the question of observability. We need some additional information on the eigenvalues:

**Lemma 3.4.** We have

\[
\lim_{k \to \pm \infty} (\omega_{k+1}^+ - \omega_k^+) = \infty.
\]

*Proof.* Since \(\omega_{-k}^+ = -\omega_k^+\), it suffices to consider the case \(k \to \infty\). Since

\[
2ck^{-3} \omega_k^+ = c + 1 + rk^{-2} + \sqrt{4acd + [c - 1 + rk^{-2}]^2},
\]

introducing the non-zero number

\[
A := \frac{c + 1 + \sqrt{4acd + (c - 1)^2}}{2c}
\]

for brevity we have

\[
\omega_k^+ = Ak^3 + O(k) \quad \text{as} \quad k \to \infty.
\]

Hence

\[
\omega_{k+1}^+ - \omega_k^+ = A [(k + 1)^3 - k^3] + O(k) = 3Ak^2 + O(k) \quad \text{as} \quad k \to \infty.
\]

Since \(A > 0\), the lemma follows. \(\square\)

**Lemma 3.5.** We have

\[
\lim_{k \to \pm \infty} (\omega_{k+1}^- - \omega_k^-) = \begin{cases} 
\infty & \text{if } ad < 1, \\
-\infty & \text{if } ad > 1, \\
\frac{r}{c(c+1)} & \text{if } ad = 1.
\end{cases}
\]

*Proof.* Since \(\omega_{-k}^- = -\omega_k^-\), it suffices to consider the case \(k \to \infty\). We have

\[
2ck^{-3} \omega_k^- = c + 1 - rk^{-2} - \sqrt{4acd + [c - 1 + rk^{-2}]^2}
\]

\[
= \frac{[c + 1 - rk^{-2}]^2 - 4acd - [c - 1 + rk^{-2}]^2}{c + 1 - rk^{-2} + \sqrt{4acd + [c - 1 + rk^{-2}]^2}}
\]

\[
= \frac{2c[2 - 2rk^{-2}] - 4acd}{c + 1 - rk^{-2} + \sqrt{4acd + [c - 1 + rk^{-2}]^2}}
\]

\[
= \frac{4c(1 - ad) - 4rk^{-2}}{c + 1 - rk^{-2} + \sqrt{4acd + [c - 1 + rk^{-2}]^2}}.
\]

If \(ad \neq 1\), then the last expression converges to the non-zero number

\[
B := \frac{4c(1 - ad)}{c + 1 + \sqrt{4acd + (c - 1)^2}}
\]

as \(k \to \pm \infty\). (It may be shown by a direct computation, although we do not need this in the present proof, that the denominators in (3.9) are non-zero for every \(k\).) Hence

\[
\omega_k^- = \frac{B}{2c} k^3 + O(k),
\]
and therefore
\[ \omega_{k+1}^- - \omega_k^- = B \left[ (k+1)^3 - k^3 \right] + O(k) = \frac{3B}{2c} k^2 + O(k), \quad k \to \pm \infty. \]
This implies the first two cases of the lemma because \( B \) and \( 1 - ad \) have the same sign.

If \( ad = 1 \), then (3.9) implies the relation
\[ \omega_k^- = \frac{-r}{c(c+1)} k + O(k^{-1}) \]
as \( k \to \pm \infty \). Hence
\[ \omega_{k+1}^- - \omega_k^- \to \frac{-r}{c(c+1)}, \quad k \to \pm \infty. \]

The rest of this section is devoted to the proof of the following

**Theorem 3.6.** Assume that
\[(3.10) \quad \omega_k^+ \neq \omega_n^+ \quad \text{and} \quad \omega_k^- \neq \omega_n^- \quad \text{whenever} \quad k \neq n. \]
Then the solutions of (3.1) have the following properties:

(i) the direct inequality
\[ \int_I \left| u(t,x) \right|^2 + \left| v(t,x) \right|^2 \, dt \ll E \]
holds for all non-degenerate bounded intervals \( I \).

(ii) if \( ad \neq 1 \), then the inverse inequality
\[(3.11) \quad E \ll \int_I \left| u(t,x) \right|^2 + \left| v(t,x) \right|^2 \, dt \]
also holds for all non-degenerate bounded intervals \( I \).

(iii) if \( ad = 1 \), then (3.11) holds for all bounded intervals \( I \) of length \( |I| > 2\pi(c+1)/r \), and it fails if \( |I| < 2\pi(c+1)/r \).

**Remark.** The assumption (3.10) is will not be needed for the proof of the direct inequality.
We will show in Lemma 3.7 below that (3.10) is satisfied for almost all \((a,c,d,r) \in (0,\infty)^4\).

We proceed in several steps.

**Proof of the direct inequality in Theorem 3.6.** Fix an arbitrary bounded interval \( I \). Writing
\[(3.12) \quad Z^+_k = \left( \begin{array}{c} z^+_{k,1} \\ z^+_{k,2} \end{array} \right) \quad \text{and} \quad Z^-_k = \left( \begin{array}{c} z^-_{k,1} \\ z^-_{k,2} \end{array} \right) \]
for brevity, using the Young inequality, applying Theorem 2.1 (ii), the following estimates yield the direct inequality:

$$
\int_I |u(t, x)|^2 + |v(t, x)|^2 \, dt = \int_I \left\| \sum_{k \in \mathbb{Z}} e^{ikx} \left[ c_k^+ e^{i\omega_k^+ t} Z_k^+ + c_k^- e^{i\omega_k^- t} Z_k^- \right] \right\|^2 \, dt
$$

$$
\leq 2 \int_I \left\| \sum_{k \in \mathbb{Z}} e^{ikx} c_k^+ e^{i\omega_k^+ t} Z_k^+ \right\|^2 \, dt + 2 \int_I \left\| \sum_{k \in \mathbb{Z}} e^{ikx} c_k^- e^{i\omega_k^- t} Z_k^- \right\|^2 \, dt
$$

$$
= 2 \int_I \left\| \sum_{k \in \mathbb{Z}} e^{ikx} c_k^+ e^{i\omega_k^+ t} z_{k,1}^+ \right\|^2 \, dt + 2 \int_I \left\| \sum_{k \in \mathbb{Z}} e^{ikx} c_k^- e^{i\omega_k^- t} z_{k,1}^- \right\|^2 \, dt
$$

$$
+ 2 \int_I \left\| \sum_{k \in \mathbb{Z}} e^{ikx} c_k^+ e^{i\omega_k^+ t} z_{k,2}^+ \right\|^2 \, dt + 2 \int_I \left\| \sum_{k \in \mathbb{Z}} e^{ikx} c_k^- e^{i\omega_k^- t} z_{k,2}^- \right\|^2 \, dt
$$

$$
\leq \sum_{k \in \mathbb{Z}} \sum_{j=1}^2 \left| e^{ikx} c_k^+ z_{k,j}^+ \right|^2 + \left| e^{ikx} c_k^- z_{k,j}^- \right|^2
$$

$$
= \sum_{k \in \mathbb{Z}} \left| e^{ikx} c_k^+ \right|^2 \| Z_k^+ \|^2 + \left| e^{ikx} c_k^- \right|^2 \| Z_k^- \|^2
$$

$$
= \sum_{k \in \mathbb{Z}} \left| c_k^+ \right|^2 \| Z_k^+ \|^2 + \left| c_k^- \right|^2 \| Z_k^- \|^2.
$$

Using (3.8) we conclude that

$$
\int_I |u(t, x)|^2 + |v(t, x)|^2 \, dt \ll E. \quad \square
$$

Proof of a weakened version of the inverse inequality in Theorem 3.6.

Fix a positive number $K$ whose value will be precised later, and consider only solutions of the form (3.6) satisfying $c_k^+ = 0$ whenever $|k| \leq K$. Using the Young inequality and applying Theorem 2.3 we have

$$
\int_I |u(t, x)|^2 + |v(t, x)|^2 \, dt = \int_I \left\| \sum_{|k| \leq K} e^{ikx} \left[ c_k^+ e^{i\omega_k^+ t} Z_k^+ + c_k^- e^{i\omega_k^- t} Z_k^- \right] \right\|^2 \, dt
$$

$$
\geq \frac{1}{2} \int_I \left\| \sum_{|k| > K} e^{ikx} \left[ c_k^+ e^{i\omega_k^+ t} Z_k^+ + c_k^- e^{i\omega_k^- t} Z_k^- \right] \right\|^2 \, dt
$$

$$
- \int_I \left\| \sum_{|k| > K} e^{ikx} \left[ c_k^+ e^{i\omega_k^+ t} (Z^+ - Z_k^+) + c_k^- e^{i\omega_k^- t} (Z^- - Z_k^-) \right] \right\|^2 \, dt
$$

$$
\geq \frac{1}{2} \int_I \left\| \sum_{|k| > K} e^{ikx} \left[ c_k^+ e^{i\omega_k^+ t} Z_k^+ + c_k^- e^{i\omega_k^- t} Z_k^- \right] \right\|^2 \, dt
$$

$$
- \delta_K \alpha \sum_{|k| > K} \left( |c_k^+|^2 + |c_k^-|^2 \right)$$
with
\[ \delta_K := \sup \left\{ \left| Z^+ - Z_k^+ \right|^2, \left| Z^- - Z_k^- \right|^2 : |k| > K \right\}. \]

Since \( Z^+ \) and \( Z^- \) are orthogonal by Lemma 3.2, using also the last part of this lemma it follows that
\[
\int_I |u(t, x)|^2 + |v(t, x)|^2 \, dt \\
\geq \frac{1}{2} \int_I \left\| \sum_{k \in \mathbb{Z}, |k| > K} e^{ikx} c_k^+ e^{i\omega_k^+ t} Z^+ \right\|^2 + \left\| \sum_{k \in \mathbb{Z}, |k| > K} e^{ikx} c_k^- e^{i\omega_k^- t} Z^- \right\|^2 \, dt \\
- \delta_K \alpha \sum_{k \in \mathbb{Z}, |k| > K} \left( |c_k^+|^2 + |c_k^-|^2 \right).
\]

Since \( \|Z^\pm\| \geq 2ac \) by Lemma 3.2, hence we infer that
\[
\int_I |u(t, x)|^2 + |v(t, x)|^2 \, dt \\
\geq 2a^2 c^2 \int_I \left\| \sum_{k \in \mathbb{Z}, |k| > K} e^{ikx} c_k^+ e^{i\omega_k^+ t} \right\|^2 + \left\| \sum_{k \in \mathbb{Z}, |k| > K} e^{ikx} c_k^- e^{i\omega_k^- t} \right\|^2 \, dt \\
- \delta_K \alpha \sum_{k \in \mathbb{Z}, |k| > K} \left( |c_k^+|^2 + |c_k^-|^2 \right).
\]

Assume that \( ad \neq 1 \). If \( K \) is sufficiently large, say \( K \geq K_0 \), then by Lemmas 3.4 and 3.5 we may apply Theorem 2.1 (iii) to conclude with some constant \( \beta > 0 \) the estimate
\[
\int_I |u(t, x)|^2 + |v(t, x)|^2 \, dt \\
\geq \beta \sum_{k \in \mathbb{Z}, |k| > K} \left( |e^{ikx} c_k^+|^2 + |e^{ikx} c_k^-|^2 \right) - \delta_K \alpha \sum_{k \in \mathbb{Z}, |k| > K} \left( |c_k^+|^2 + |c_k^-|^2 \right) \\
= (\beta - \delta_K \alpha) \sum_{k \in \mathbb{Z}, |k| > K} \left( |c_k^+|^2 + |c_k^-|^2 \right) \\
\asymp (\beta - \delta_K \alpha) E.
\]

In the last step we used (3.8). Since \( \delta_K \to 0 \) as \( K \to \infty \), choosing a sufficiently large \( K \geq K_0 \) we have \( \beta - \delta_K \alpha > 0 \), so that, under the assumption \( ad \neq 1 \) the inverse inequality holds for all functions of the form (3.6) satisfying \( c_k^\pm = 0 \) whenever \( |k| \leq K \).

If \( ad = 1 \), then by Lemma 3.5 we may repeat the last reasoning for every bounded interval of length
\[
|I| > \frac{2\pi}{r} = \frac{2\pi c(c + 1)}{r},
\]
\( \square \).
Now we show that Theorem 3.6 holds for almost all choices of the parameters $a, c, d, r$:

**Lemma 3.7.** For almost every quadruple $(a, c, d, r) \in (0, \infty)^4$, we have $\omega_k^+ \neq \omega_n^+$ and $\omega_k^- \neq \omega_n^-$ whenever $k \neq n$.

**Proof.** Setting

$$I_k := 4acdk^4 + [r + (c - 1)k^2]^2 = r^2 + (2c - 2)k^2r + [(c - 1)^2 + 4acd]k^4$$

for brevity, in case $\omega_k^+ = \omega_n^+$ we deduce from (3.3) that

$$(c + 1)(k^3 - n^3) - (k - n)r = n\sqrt{I_n} - k\sqrt{I_k}.$$

Taking the square of both sides hence we get

$$(3.14)\quad 2nk\sqrt{I_n}\sqrt{I_k} = nkr^2 + \left[2c(n^4 + k^4) - (c + 1)nk(n^2 + k^2)\right]r + \alpha_1$$

with a constant $\alpha_1$ not depending on $r$. Taking its square again, we get after some simplification the equation

$$(3.15)\quad 2c(n - k)(n^3 - k^3)r^3 + \alpha_2r^2 + \alpha_3r + \alpha_4 = 0$$

with suitable constants $\alpha_2, \alpha_3, \alpha_4$, independent of $r$.

Similarly, in case $\omega_k^- = \omega_n^-$ we deduce from (3.3) instead of (3.13) the equality

$$(c + 1)(k^3 - n^3) - (k - n)r = k\sqrt{I_k} - n\sqrt{I_n},$$

and then the same equalities (3.14) and (3.15).

For any fixed $(a, c, d) \in (0, \infty)^3$ and for any $(k, n) \in \mathbb{Z}^2$ with $k \neq n$ the polynomial equation (3.15) vanishes for at most three values of $r$. Therefore for any fixed $(a, c, d) \in (0, \infty)^3$, $\omega_k^+ \neq \omega_n^+$ and $\omega_k^- \neq \omega_n^-$ for all $k \neq n$ for all but countable many values of $r$. The lemma follows by applying Fubini’s theorem. $\square$

**Proof of the inverse inequality in Theorem 3.6.** Thanks to Lemmas 3.4, 3.5 and 3.7 and the preceding two estimates we may apply Theorem 2.5. $\square$

### 4. Pointwise controllability

In this section we study the pointwise controllability of the following system

$$\begin{align*}
\begin{cases}
 u_t + u_{xxx} + av_{xxx} &= f(t)\delta_{x_0} & \text{in} & & \mathbb{R} \times \mathbb{T}, \\
 v_t + \frac{r}{c}v_x + \frac{1}{c}v_{xxx} + \frac{d}{c}u_{xxx} &= g(t)\delta_{x_0} & \text{in} & & \mathbb{R} \times \mathbb{T}, \\
 u(0) = u_0 & \text{and} & v(0) = v_0,
\end{cases}
\end{align*}$$

(4.1)

where $a, c, d, r$ are given positive constants, $\delta_{x_0}$ denotes the Dirac delta function centered in a given point $x_0 \in \mathbb{T}$, and $f, g$ are the control functions.

We will prove the following
Theorem 4.1. Fix $x_0 \in \mathbb{T}$ arbitrarily, and choose $a,c,d,r$ satisfying (3.10). Set

$$T_0 := \begin{cases} 0 & \text{if } ad \neq 1, \\ 2\pi c(c + 1)/r & \text{if } ad = 1. \end{cases}$$

Given $T > T_0$ arbitrarily, for every $(u_0, v_0), (u_T, v_T) \in H$ there exist control functions $f, g \in L^2_{\text{loc}}(\mathbb{R})$ such that the solution of (4.1) satisfies the final conditions

$$u(T) = u_T \quad \text{and} \quad v(T) = v_T.$$ 

Remark. In case $ad = 1$ the system is not controllable for any $T < T_0$. This follows from Theorem 3.6 and from a general theorem of duality between observability and controllability; see, e.g., [12].

By a general argument of control theory, it suffices to consider the special case of null controllability, i.e., the case where $u_T = v_T = 0$.

We prove this by applying the Hilbert Uniqueness Method (HUM) of J.-L. Lions [24] as follows.

Fix $(u_0, v_0) \in H$ arbitrarily. Choose $(\varphi_0, \psi_0) \in D(A)$ arbitrarily, and solve the homogeneous adjoint system

$$\begin{cases} \varphi_t + \varphi_{xxx} + \frac{2}{c} \psi_{xxx} = 0 & \text{in } \mathbb{R} \times \mathbb{T}, \\ \psi_t + \frac{r}{c} \psi_x + a\varphi_{xxx} + \frac{1}{c} \psi_{xxx} = 0 & \text{in } \mathbb{R} \times \mathbb{T}, \\ \varphi(0) = \varphi_0 \quad \text{and} \quad \psi(0) = \psi_0. \end{cases}$$

Then solve the non-homogeneous problem with zero final states (instead of the non-zero initial states in (4.1)) by applying the controls

$$f := -\varphi(\cdot, x_0) \quad \text{and} \quad g := -\psi(\cdot, x_0):$$

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = -\varphi(\cdot, x_0)\delta_{x_0} & \text{in } \mathbb{R} \times \mathbb{T}, \\ v_t + \frac{r}{c} v_x + \frac{1}{c} u_{xxx} + \frac{2}{c} u_{xxx} = -\psi(\cdot, x_0)\delta_{x_0} & \text{in } \mathbb{R} \times \mathbb{T}, \\ u(T) = v(T) = 0. \end{cases}$$

If $(u(0), v(0))$ happens to be equal to $(u_0, v_0)$, then the solution of (4.1) with the controls (4.3) will satisfy the final conditions of the theorem by the uniqueness (to be proven below) of the solutions of (4.1) and (4.4).

Therefore the theorem will be completed by showing that the range of the map

$$\Lambda(\varphi_0, \psi_0) := (u(0), v(0))$$

contains $H$.

Now we make this approach precise. We start by defining the solutions of (4.1). We begin with a formal computation. If $(u, v)$ solves
(4.1) and \((\varphi, \psi)\) solves (4.2), then we obtain for each fixed \(T \in \mathbb{R}\) the following equalities by integration by parts:

\[
0 = \int_0^T \int_T^0 \left( \varphi_t + \varphi_{xxx} + \frac{d}{c} \psi_{xxx} \right) \,dx \,dt \\
+ \int_0^T \int_T^0 \left( \psi_t + \frac{r}{c} \psi_x + a \varphi_{xxx} + \frac{1}{c} \psi_{xxx} \right) \,dx \,dt \\
= \left[ \int_T^0 w \varphi + v \psi \,dx \right]_0^T \\
- \int_0^T \int_T^0 (u_t + u_{xxx} + a v_{xxx}) \varphi \,dx \,dt \\
- \int_0^T \int_T^0 \left( v_t + \frac{r}{c} v_x + \frac{1}{c} v_{xxx} + \frac{d}{c} u_{xxx} \right) \psi \,dx \,dt \\
= \left[ \int_T^0 w \varphi + v \psi \,dx \right]_0^T - \int_0^T f(t) \varphi(t,x_0) + g(t) \psi(t,x_0) \,dt.
\]

This may be rewritten in the form

\[
(4.5) \quad (((u(T), v(T)), (\varphi(T), \psi(T)))_H \\
= (((u_0, v_0), (\varphi_0, \psi_0))_H + ((f, g), (\varphi(\cdot, x_0), \psi(\cdot, x_0))))_{G_T},
\]

where \(G_T\) denotes the Hilbert space \(L^2(0, T) \times L^2(0, T)\) for \(T > 0\) and \(L^2(T, 0) \times L^2(T, 0)\) for \(T < 0\). This identity leads to the following definition:

**Definition.** By a solution of (4.1) we mean a function \((u, v) \in C_w(\mathbb{R}, H)\) satisfying (4.5) for all \(T \in \mathbb{R}\) and \((\varphi_0, \psi_0) \in H\).

The subscript \(w\) indicates that \(H\) is endowed here with the weak topology. The definition is justified by the following

**Theorem 4.2.** Given any initial data \((u_0, v_0) \in H\) and control functions \(f, g \in L^2_{\text{loc}}(\mathbb{R})\), the system (4.1) has a unique solution, and the linear map

\[
(u_0, v_0, f, g) \mapsto (u, v)
\]

is continuous for the indicated topologies.

**Remark.** By the time invariance of the system (4.1) the theorem remains valid if we impose final conditions instead of initial conditions; hence it also proves the well posedness of (4.4).

**Proof.** Denoting the right hand side of (4.5) by \(L_T(\varphi_0, \psi_0)\) for each fixed \(T \in \mathbb{R}\), it follows from Theorem 3.3 and the direct inequality in Theorem 3.6 that \(L_T\) is a continuous linear functional of \((\varphi_0, \psi_0)\), and therefore also of \((\varphi(T), \psi(T))\). This proves the existence of a unique couple \((u(T), v(T)) \in H\) satisfying (4.5).
The weak continuity of the solution follows by observing that for any fixed \((\varphi_0, \psi_0) \in H\) the right hand side of (4.5) is continuous in \(T\) by the continuity of the primitives of Lebesgue integrable real functions.

Using Theorem 3.3 again, we may also infer from (4.5) for each \(T > 0\) the estimates
\[
\|(u, v)\|_{L^\infty(-T,T; H)} \leq \left\{ \|(u_0, v_0)\|_H + \|(f, g)\|_{L^2(-T,T)} \right\}
\]
for all \(T > 0\); this proves the continuous dependence of the solution on the data. □

**Remark.** Since we have only used the direct inequality in Theorem 3.6, Theorem 4.2 holds without the assumption (3.10) on the eigenvalues.

Now we turn back to the proof of Theorem 4.1. It remains to show that the range of the map \(\Lambda\) contains \(H\).

Indeed, it is a continuous linear map \(\Lambda : H \to H\) by Theorems 3.3 and 4.2. Furthermore, we infer from (4.5) that
\[
(\Lambda(\varphi_0, \psi_0), (\varphi_0, \psi_0))_H = \int_0^T |\varphi(t, x_0)|^2 + |\psi(t, x_0)|^2 \, dt,
\]
and then from Theorem 3.6 that
\[
(\Lambda(\varphi_0, \psi_0), (\varphi_0, \psi_0))_H \geq \varepsilon \|(\varphi_0, \psi_0)\|^2_H
\]
for all \((\varphi_0, \psi_0) \in H\) with a suitable positive constant \(\varepsilon\).

Applying the Lax–Milgram theorem we conclude that \(\Lambda\) is an isomorphism of \(H\) onto itself; in particular, it is onto.

5. **Pointwise stabilizability**

Following [22, Chapter 2] first we recall some general abstract results. Consider the evolutionary problem
\[
(5.1) \quad U' = AU, \quad U(0) = U_0,
\]
(we use the notation \(U'\) for the time derivative of \(U\)) where

(i) \(A\) is a skew-adjoint linear operator in a Hilbert space \(\mathcal{H}\), having a compact resolvent.

Then \(A\) generates a strongly continuous group of automorphisms \(e^{tA}\) in \(\mathcal{H}\), and for each \(U_0 \in \mathcal{H}\) the problem (5.1) has a unique continuous solution \(U : \mathbb{R} \to \mathcal{H}\), satisfying the equality
\[
\|U(t)\| = \|U_0\| \quad \text{for all} \quad t \in \mathbb{R}.
\]

Now let \(B\) be another linear operator, defined on some linear subspace \(D(B)\) of \(\mathcal{H}\) with values in another Hilbert space \(\mathcal{G}\), satisfying the following two hypotheses:

(ii) \(D(A) \subset D(B)\), and there exists a constant \(c\) such that
\[
\|BU_0\|_\mathcal{G} \leq c \|AU_0\|_\mathcal{H} \quad \text{for all} \quad U_0 \in D(A);
\]
(iii) There exist a non-degenerate bounded interval $I$ and a constant $c_I$ such that the solutions of (5.1)
\[ \|BU\|_{L^2(I;G)} \leq c_I \|U_0\|_H \quad \text{for all} \quad U_0 \in D(A). \]
The operator $B$ is usually called an observability operator: We may think that we can observe only $BU$ and not the whole solution $U$.

Under the assumptions (i), (ii), (iii) we may define the solutions of the dual problem
\[ V' = -A^*V + B^*W, \quad V(0) = V_0, \]
by transposition, were $\mathcal{H}', \mathcal{G}'$ denote the dual spaces of $\mathcal{H}, \mathcal{G}$, and $A^*$, $B^*$ denote the adjoints of $A$ and $B$.

The operator $B^*$ is usually called a controllability operator: we may think that we can act on the system by choosing a control $W$.

Motivated by a formal computation, by a solution of (5.2) we mean a function $V : \mathbb{R} \to \mathcal{H}'$ satisfying the identity
\[ \langle V(S), U(S) \rangle_{\mathcal{H}', \mathcal{H}} = \langle V_0, U_0 \rangle_{\mathcal{H}', \mathcal{H}} + \int_0^s \langle W(t), BU(t) \rangle_{\mathcal{G}', \mathcal{G}} \, dt \]
for all $U_0 \in \mathcal{H}$ and for all $s \in \mathbb{R}$. Then for every $V_0 \in \mathcal{H}'$ and $W \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{G}')$, (5.2) has a unique solution. Moreover, the function $V : \mathbb{R} \to \mathcal{H}'$ is weakly continuous.

The Hilbert Uniqueness Theorem states that if the inverse inequality of (iii) also holds:

(iv) There exist a bounded interval $I'$ and a constant $c'$ such that the solutions of (5.1) satisfy the inequality
\[ \|U_0\|_H \leq c' \|BU\|_{L^2(I';G)} \quad \text{for all} \quad U_0 \in D(A), \]
then the system (5.2) is exactly controllable in the following sense:

\begin{theorem}
Assume (i), (ii), (iii), (iv), and let $T > |I'|$ (the length of $I'$). Then for all $V_0, V_1 \in \mathcal{H}'$ there exists a function $W \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{G}')$ such that the solution of (5.2) satisfies $V(T) = V_1$.
\end{theorem}

We may of course assume that $W$ vanishes outside the interval $(0, T)$.

Under the assumptions of the theorem we may also construct feedback controls yielding arbitrarily fast decay rates:

\begin{theorem}
Assume (i), (ii), (iii), (iv), and fix $\omega > 0$ arbitrarily. There exists a bounded linear map $F : \mathcal{H}' \to \mathcal{G}'$ and a constant $M > 0$ such that the problem
\[ V' = -A^*V + B^*FV, \quad V(0) = V_0 \]
has a unique weakly continuous solution $V : \mathbb{R} \to \mathcal{H}'$, and the solutions satisfy the estimates
\[ \|V(t)\|_{\mathcal{H}'} \leq M \|V_0\|_{\mathcal{H}'} e^{-\omega t} \]
for all $V_0 \in \mathcal{H}'$ and $t \geq 0$.
\end{theorem}
Let us observe that Theorems 5.1 and 5.2 have the same assumptions. These assumptions have been verified during the proof of Theorem 1.1. Therefore we may also apply Theorem 5.2 for the Gear–Grimshaw system, and Theorem 1.3 follows.

6. Use of one control

In this section we establish a variant of Theorem 3.6 when we observe only one of the functions \( u(\cdot, x_0) \) and \( v(\cdot, x_0) \).

Such observations do not allow us to determine completely the initial data in (3.1). Indeed, if \((u, v)\) solves (3.1), then for any constant \( c \) the couple \((u, v + c)\) also solves (3.1) with some other initial data, so that the observation of the component \( u \) may allow to determine \( v \) up to an arbitrary additive constant. An analogous situation occurs by observing \( v \).

Theorem 6.2 and Corollary 6.3 below will show that up to this indeterminacy the determination of the solutions is possible by observing only one component.

Since the solutions of (3.1) are given by the formulas

\[
\begin{align*}
u(t) &= \sum_{k \in \mathbb{Z}} \left( c_k^+ e^{i\omega_k^+ t} z_{k,1}^+ + c_k^- e^{i\omega_k^- t} z_{k,1}^- \right) e^{ikx}, \\
v(t) &= \sum_{k \in \mathbb{Z}} \left( c_k^+ e^{i\omega_k^+ t} z_{k,2}^+ + c_k^- e^{i\omega_k^- t} z_{k,2}^- \right) e^{ikx}
\end{align*}
\]

(we use the notation (3.12)), we need an Ingham type theorem for the family \( \{ \omega_k^\pm : k \in \mathbb{Z} \} \). It does not have a uniform gap, because \( \omega_0^+ = \omega_0^- = 0 \) and because \( \omega_k^\pm \) may be close to \( \omega_n^\pm \) for many couples \((k, n)\), but it satisfies the weakened gap condition of Theorem 2.2 with \( M = 2 \).

Given a positive number \( \varepsilon \) we consider in the set

\[
\Omega := \{ \omega_k^\pm : k \in \mathbb{Z} \}
\]

the equivalence relation

\[
x \sim y \iff |x - y| < \varepsilon.
\]

**Lemma 6.1.**

(i) We have

\[
|z_{k,j}^\pm| < 1, \quad j = 1, 2.
\]

(ii) For almost every quadruple \((a, c, d, r) \in (0, \infty)^4\), we have

\[
\omega_k^+ \neq \omega_n^+ \quad \text{and} \quad \omega_k^- \neq \omega_n^- \quad \text{whenever} \quad k \neq n,
\]

and

\[
\omega_k^+ \neq \omega_n^- \quad \text{for all} \quad k, n \in \mathbb{Z}, \quad \text{except if} \quad k = n = 0.
\]

(iii) Assume (6.2) and (6.3). If \( \varepsilon \) sufficiently small, then no equivalence class has more than two elements.
Proof. (i) Readily follows from the explicit expression (3.4) of these vectors.

(ii) In view of Lemma 3.7 we only need to consider the property (6.3). For this we adopt the proof of Lemma 3.7 as follows.

If \( \omega^+_k = \omega^-_n \) for some \( k, n \in \mathbb{Z} \), then we deduce from (3.3) the equality
\[
(c + 1)(k^3 - n^3) - (k - n)r = -n\sqrt{I_n} - k\sqrt{I_k}
\]
instead of (3.13), and then the equality
\[
-nk\sqrt{I_n}\sqrt{I_k} = nkr^2 + [2c(n^4 + k^4) - (c + 1)nk(n^2 + k^2)] r + \alpha_1
\]
instead of (3.14). Taking its square we arrive at the same equation (3.15) as before.

(iii) For any fixed \( \epsilon > 0 \), by Lemmas 3.4 and 3.5 there exists a sufficiently large positive integer \( K \) such that
\[
|\omega^+_k - \omega^-_n| \geq 2\epsilon \quad \text{and} \quad |\omega^-_k - \omega^-_n| \geq 2\epsilon
\]
whenever \( |k|, |n| > K \) and \( k \neq n \). Then each equivalence class in the restricted set
\[
\{ \omega^+_k : |k| > K \}
\]
has at most two elements. Indeed, if two elements are equivalent, then they have to belong to the different families \( \{ \omega^+_k \} \) and \( \{ \omega^-_k \} \), say
\[
|\omega^+_k - \omega^-_n| < \epsilon.
\]
Then we infer from our choice of \( K \) that
\[
|\omega^+_k - \omega^-_m| > \epsilon \quad \text{for all} \quad m \neq n
\]
and
\[
|\omega^-_m - \omega^-_n| > \epsilon \quad \text{for all} \quad m \neq k,
\]
so that no other exponent is equivalent to \( \omega^+_k \) and \( \omega^-_n \).

This property remains valid if we change \( \epsilon \) to a smaller positive value. Indeed, each one-point equivalence class remains the same, while the others either remain the same or they split into two one-point equivalence classes.

Next we observe that each element of \( \Omega \) is isolated. Therefore, if we diminish \( \epsilon \) so as to satisfy the finite number of inequalities
\[
dist(\omega^+_k, \Omega \setminus \{ \omega^+_k \}) > \epsilon \quad \text{for} \quad k = 0, \pm 1, \ldots, \pm K,
\]
and
\[
dist(\omega^-_k, \Omega \setminus \{ \omega^-_k \}) > \epsilon \quad \text{for} \quad k = 0, \pm 1, \ldots, \pm K,
\]
then \( \{ \omega^+_k \} \) and \( \{ \omega^-_k \} \) will be one-point equivalence classes for each \( k = 0, \pm 1, \ldots, \pm K \). (Let us observe that the equality \( \omega^+_0 = \omega^-_0 = 0 \) does not contradict these properties because \( \omega^+_0 \) and \( \omega^-_0 \) are the same element of \( \Omega \).) \( \Box \)
Under the conditions of Lemma 6.1 we may rewrite the solutions (6.1) of (3.1) as follows.

Whenever \( \omega^+_n \sim \omega^-_n \), we rewrite the corresponding terms in the form

\[
(6.4) \quad c_k^+ e^{i \omega^+_n t} z_{k,j}^+ + c_n^- e^{i \omega^-_n t} z_{n,j}^- = a_{k,j}^+ z_{k,j}^+ e^{i \omega^+_n t} + a_{n,j}^- z_{n,j}^- \frac{e^{i \omega^+_n t} - e^{i \omega^-_n t}}{\omega^+_k - \omega^-_n}
\]

for \( j = 1, 2 \), and we write

\[
(6.5) \quad e_k^+(t) := e^{i \omega^+_n t}, \quad e_n^-(t) := \frac{e^{i \omega^+_n t} - e^{i \omega^-_n t}}{\omega^+_k - \omega^-_n}.
\]

For all other exponents (in particular, for \( \omega^+_0 = \omega^-_0 = 0 \)) we set

\[
e_k^+(t) := e^{i \omega^+_n t} \quad \text{and} \quad a_{k,j}^+ := c_k^+.
\]

We have thus instead of (6.1) the following representation:

\[
(6.6) \quad u(t) = \sum_{k \in \mathbb{Z}} \left( a_{k,1}^+ z_{k,1}^+ e_k^+(t) + a_{k,1}^- z_{k,1}^- e_k^-(t) \right) e^{ikx},
\]

\[
v(t) = \sum_{k \in \mathbb{Z}} \left( a_{k,2}^+ z_{k,2}^+ e_k^+(t) + a_{k,2}^- z_{k,2}^- e_k^-(t) \right) e^{ikx}.
\]

Using this representation we may state our theorem, where we use the notation \( \mathbb{Z}^* := \mathbb{Z} \setminus \{0\} \):

**Theorem 6.2.** Assume (6.2) and (6.3), and fix \( x_0 \in \mathbb{T} \) arbitrarily. Then the solutions of (3.1) have the following properties:

(i) the direct inequalities

\[
\int_I |u(t, x_0)|^2 \, dt \ll |a^+_{0,1} + a^-_{0,1}|^2 + \sum_{k \in \mathbb{Z}^*} \left( |a^+_{k,1}|^2 + |a^-_{k,1}|^2 \right)
\]

and

\[
\int_I |v(t, x_0)|^2 \, dt \ll |a^+_{0,2} - a^-_{0,2}|^2 + \sum_{k \in \mathbb{Z}^*} \left( |a^+_{k,2}|^2 + |a^-_{k,2}|^2 \right)
\]

hold for all non-degenerate bounded intervals \( I \).

(ii) if \( ad \neq 1 \), then the inverse inequalities

\[
(6.7) \quad |a^+_{0,1} + a^-_{0,1}|^2 + \sum_{k \in \mathbb{Z}^*} \left( |a^+_{k,1}|^2 + |a^-_{k,1}|^2 \right) \ll \int_I |u(t, x_0)|^2 \, dt
\]

and

\[
(6.8) \quad |a^+_{0,2} - a^-_{0,2}|^2 + \sum_{k \in \mathbb{Z}^*} \left( |a^+_{k,2}|^2 + |a^-_{k,2}|^2 \right) \ll \int_I |v(t, x_0)|^2 \, dt
\]

also hold for all non-degenerate bounded intervals \( I \).

(iii) if \( ad = 1 \), then (6.7) and (6.8) hold for all bounded intervals \( I \) of length \( |I| > 2\pi c(c+1)/r \), and they fail if \( |I| < 2\pi c(c+1)/r \).
**Proof of Theorem 6.2.** Thanks to Lemmas 3.4, 3.5 and 6.1 the theorem follows by applying Theorem 2.2 with \( M = 2 \).

For \( k = 0 \) we also use the relations (see (3.5) and (6.5))
\[
a_{0,1}^+ z_{0,1}^+ e_{0}^+(t) + a_{0,1}^- z_{0,1}^- e_{0}^-(t) = 2ac(a_{0,1}^+ + a_{0,1}^-)
\]
and
\[
a_{0,2}^+ z_{0,2}^+ e_{0}^+(t) + a_{0,2}^- z_{0,2}^- e_{0}^-(t) = \sqrt{4acd(a_{0,2}^+ - a_{0,2}^-)}.
\]

It remains to show that the upper density of the family \( \{ \omega_k^\pm \} \) is equal to 0 if \( ad \neq 1 \), and equal to \( c(c + 1)/r \) if \( ad = 1 \). Using the general equality \( D^+ = 1/\gamma_\infty \), it follows from Lemmas 3.4 and 3.5 that
\[
D^+(\{ \omega_k^+ \}) = 0
\]
and
\[
D^+(\{ \omega_k^- \}) = \begin{cases} 0 & \text{if } ad \neq 1, \\ c(c + 1)/r & \text{if } ad = 1. \end{cases}
\]
Since
\[
D^+(\{ \omega_k^\pm \}) = D^+(\{ \omega_k^+ \}) + D^+(\{ \omega_k^- \}),
\]
this implies our claim. \( \square \)

We infer from Theorem 6.2 the following uniqueness property:

**Corollary 6.3.** Assume (6.2) and (6.3), and set
\[
T_0 := \begin{cases} 0 & \text{if } ad \neq 1, \\ 2\pi c(c + 1)/r & \text{if } ad = 1. \end{cases}
\]

Fix \( x_0 \in \mathbb{T} \) arbitrarily, consider the solution of (3.1) and an interval \( I \) of length \( |I| > T_0 \).

(i) If \( u(t, x_0) = 0 \) for all \( t \in I \), then \( u = 0 \) and \( v \) is an arbitrary constant function.

(ii) If \( v(t, x_0) = 0 \) for all \( t \in I \), then \( v = 0 \) and \( u \) is an arbitrary constant function.

**Proof.** If \( u(t, x_0) = 0 \) for all \( t \in I \), then we infer from the estimate of Theorem 6.2 the equalities
\[
a_{0,1}^+ + a_{0,1}^- = 0, \quad \text{and} \quad a_{k,1}^\pm = 0 \quad \text{for all} \quad k \in \mathbb{Z}^*.
\]
In view of (6.4) this is equivalent to the relations
\[
e_{0}^+ + e_{0}^- = 0, \quad \text{and} \quad c_k^\pm = 0 \quad \text{for all} \quad k \in \mathbb{Z}^*.
\]
Using (3.5) we conclude that
\[
u(t, x) = \sqrt{4acd(c_0^+ - c_0^-)},
\]
i.e., \( u = 0 \) and \( v \) is an arbitrary constant function.
Similarly, if \( u(t, x_0) = 0 \) for all \( t \in I \), then we obtain that
\[
c_0^+ - c_0^- = 0, \quad \text{and} \quad c_k^\pm = 0 \quad \text{for all} \quad k \in \mathbb{Z}^*.
\]
This implies that \( v = 0 \) and \( u \) is an arbitrary constant function.
\[\square\]

We end this paper by proving two variants of Theorem 4.1 where we apply only one control.

Let us observe that if \( f = 0 \) in (4.1), then \( \int_T u(t, x) \, dx \) does not depend on \( t \in \mathbb{R} \) because
\[
\frac{d}{dt} \int_T u \, dx = - \int_T u_{xxx} + av_{xxx} \, dx = 0
\]
by integration by parts. It follows that by using only the control function \( g \) in (4.1), if a state \((u_0, v_0) \in H\) may be driven to \((u_T, v_T) \in H\) in time \( T \), then
\[
\int_T u_0 \, dx = \int_T u_T \, dx. \tag{6.9}
\]

Similarly, if \( g = 0 \) in (4.1), then \( \int_T v(t, x) \, dx \) does not depend on \( t \in \mathbb{R} \) because
\[
\frac{d}{dt} \int_T v \, dx = - \int_T r_cv_x + \frac{1}{c}v_{xxx} + \frac{d}{c}u_{xxx} \, dx = 0.
\]
It follows that by using only the control function \( f \) in (4.1), if a state \((u_0, v_0) \in H\) may be driven to \((u_T, v_T) \in H\) in time \( T \), then
\[
\int_T v_0 \, dx = \int_T v_T \, dx. \tag{6.10}
\]

Applying the method of “contrôlabilité exacte élargie” of Lions [24, p. 95], we prove that these conditions are also sufficient for the controllability if the time is large enough.

**Theorem 6.4.** Fix \( x_0 \in \mathbb{T} \) arbitrarily, and choose \( a, c, d, r \) satisfying (3.10), and set
\[
T_0 := \begin{cases} 
0 & \text{if } ad \neq 1, \\
2\pi c(c + 1)/r & \text{if } ad = 1.
\end{cases}
\]

Furthermore, fix \( T > T_0 \) and \((u_0, v_0), (u_T, v_T) \in H\) arbitrarily.

(i) There exists a control function \( f \in L^2_{loc} (\mathbb{R}) \) such that the solution of (4.1) with \( g = 0 \) satisfies the final conditions
\[
u(T) = u_T \quad \text{and} \quad v(T) = v_T
\]
if and only if (6.10) is satisfied.

(ii) There exists a control function \( g \in L^2_{loc} (\mathbb{R}) \) such that the solution of (4.1) with \( f = 0 \) satisfies the final conditions
\[
u(T) = u_T \quad \text{and} \quad v(T) = v_T
\]
if and only if (6.9) is satisfied.
Proof. The two cases being analogous, we only consider (i).

We have already shown the necessity of the condition (6.10). It remains to prove the null controllability of all initial data \((u_0, v_0)\) belonging to the closed linear subspace

\[
\tilde{H} := \left\{ (u_0, v_0) \in H : \int_T u_0 \, dx = 0 \right\}.
\]

Now we repeat the proof of Theorem 4.1 by modifying the definition of the operator \(\Lambda\): in (4.3) and (4.4) we change \(g = -\psi(\cdot, x_0)\) to 0. We obtain a continuous linear map \(\Lambda : \tilde{H} \to \tilde{H}\), and the hypotheses of the Lax–Milgram theorem are satisfied by Theorem 6.2.

\[\square\]

Remark. It follows from Theorem 6.2 that the value \(T_0\) is optimal for the validity of Corollary 6.3 and Theorem 6.4.

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References


