Federal University of Pernambuco
Department of Physics

General Doctorate Examination
Second Semester, 2014

Classical Electrodynamics

05/08/2014 - 09:00 to 12:00 h

(Choose 3 of the 4 problems)
**Problem 1 – Electrostatics**

1. A dielectric sphere of radius $a$ and dielectric constant $\epsilon$ is in the presence of an external uniform electric field $\mathbf{E}_0 = \hat{z} E_0$ as shown in Fig. A. The resulting electric potential is

$$\begin{align*}
\Phi_1(r, \theta) &= Ar \cos \theta \quad ; \quad r < a \\
\Phi_2(r, \theta) &= Br \cos \theta + \frac{C}{r^2} \cos \theta \quad ; \quad r \geq a,
\end{align*}$$

where $A$, $B$ and $C$ are constants.

(a) [50%] Use the adequate boundary conditions to determine the coefficients $A$, $B$ e $C$.

(b) [20%] Determine the polarization $\mathbf{P}$ inside the sphere and calculate the dipole moment of this distribution.

(c) [30%] Suppose now that the external electric field at the position of the sphere is created by an infinitely long wire with a charge density $\lambda$ at a distance $R >> a$, as shown in Fig. B. Suppose also that the polarization inside the sphere may be considered approximately constant and calculate the force $\mathbf{F}$ on the sphere, which is given by

$$\mathbf{F} = \int (\mathbf{P} \cdot \nabla) \mathbf{E}_{ext} \, dV.$$
Problem 2 – Magnetostatics

2. A ring with positive electric charge $Q$ uniformly distributed is located at $x^2 + y^2 = R^2$ and rotates with angular velocity $\omega = \omega_0 \hat{z}$.

(a) [15%] Determine the magnetic field $\mathbf{B}_{\text{ring}}$ generated by the ring at the $z$ axis.

(b) [15%] A current line $I_0$ is located at $y = 0; z = H$. Determine the magnetic field $\mathbf{B}_{\text{wire}}$ generated by this current at the origin. The current is positive in the $x$ direction.

Suppose now that $H >> R$

(c) [30%] Determine the torque on the ring.

(d) [30%] Find the interaction energy between the wire and the ring.

(e) [10%] Obtain the dipole moment vector in the orientation of minimum energy.
PROBLEM 3 – ELECTRIC CIRCUITS

3. A long and straight wire carries a slowly varying current, returning through a coaxial conductor of radius \( r = a \), as shown in the figure below. The current is therefore uniform through all the wire and given by \( I(t) = I_0 \cos(\omega t) \). The induced electric field is longitudinal, parallel to the wire.

(a) [40%] Assuming the field is zero for \( r \to \infty \), obtain \( \mathbf{E}(r, t) \)
(b) [30%] Calculate the displacement current density \( \mathbf{J}_D(r, t) \)
(c) [20%] Determine the relation between \( I_D \) and \( I \).
(d) [10%] Suppose that the diameter of the cylinder is \( a = 2.0 \text{ mm} \). For what frequency is \( I_D = 10^{-6} I_0 \)?

\[ I(t) \]

\[ I(t) \]

\[ \mathbf{r} \]

\[ a \]
Problem 4 – Propagation of Electromagnetic Waves

4. The Earth’s ionosphere may be described as a region of uniform free electron density \( N_e \approx 10^5 \text{ elétrons/cm}^3 \), which starts at an altitude of about 100 km. The dielectric constant of this region may be approximated by

\[
\varepsilon(\omega) = \varepsilon_0 \left[ 1 - \frac{\omega_P^2}{\omega (\omega + i \gamma)} \right] \approx \varepsilon_0 \left[ 1 - \frac{\omega_P^2}{\omega^2} \right] \quad \text{where} \quad \omega_P^2 = \frac{N_e e^2}{\varepsilon_0 m_e},
\]

and \( \gamma \approx 10^5 \text{ rad/s} \) is the relaxation rate due to electron collisions.

(a) [10%] Estimate the magnitude of the plasma frequency \( \omega_P \), verifying that \( \omega_P \gg \gamma \), and therefore justifying the approximation in the equation above. Determine the frequency \( \omega \) for which the dielectric constant will be half of its value for vacuum.

(b) [30%] The propagation of an Electromagnetic wave in the ionosphere is determined by the refractive index (which may be complex)

\[
\tilde{n}(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} = n_r(\omega) + i \kappa(\omega).
\]

The reflectivity for a wave which incides perpendicularly on an interface between vacuum and a medium with refractive index \( \tilde{n} \) is

\[
R(\omega) = \left| \frac{\tilde{n}(\omega) - 1}{\tilde{n}(\omega) + 1} \right|^2.
\]

Obtain \( \tilde{n}(\omega) \) and \( R(\omega) \) for (i) \( \omega > \omega_P \) and (ii) \( \omega < \omega_P \).

(c) [30%] The propagation factor of a plane monochromatic wave travelling in the \( \hat{z} \) direction is \( \exp \left[ i \left( \tilde{n}(\omega) \frac{c}{\omega} \right) z - \omega t \right] \). Show that the attenuation of the intensity of the wave is given by

\[
\frac{dI(z)}{dz} = -\alpha I(z) \quad \text{with} \quad \alpha = \frac{2 \omega}{c} \kappa,
\]

and calculate the penetration depth \( \alpha^{-1} \) for a wave with frequency \( \omega = \omega_P / 2 \).

(d) [30%] The phase velocity and group velocity are, respectively,

\[
v_f(\omega) = \frac{c}{n_r(\omega)} \quad , \quad v_g(\omega) = \frac{c}{d[n_r(\omega)]/d\omega}.
\]

Calculate the phase and group velocities for \( \omega > \omega_P \) and comment on these results.
Useful relations

| $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ | $\nabla \cdot \mathbf{B} = 0$ |
| $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}$ | $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \mathbf{E}$ |
| $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E}$ | $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$ |

Physical constants

| $\mu_0 = 4\pi \times 10^{-7} \text{ A} \cdot \text{m}$ | $\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$ |
| $c = 3,0 \times 10^8 \text{ m/s}$ | $m_e = 9,1 \times 10^{-31} \text{ kg}$ |