Federal University of Pernambuco
Department of Physics

General Doctorate Examination
Second Semester, 2014

Quantum Mechanics

07/08/2014, 09:00 – 12:00 h

(Choose 3 of the 4 problems)
Problem 1 – Particle in one-dimensional well

Consider the following one-dimensional system

\[
\begin{array}{c}
V(x) \\
\end{array}
\]

The electron is in the eigenstate of this system with energy 1.3 eV.

(a) [30%] Find the general form of the wave function (but you do not need to calculate the normalization constant), determining the numerical value of the wave vector or of the decay constant, as applicable, for the regions \(-L \leq x \leq 0\) and \(x > 0\).

(b) [30%] Use the boundary conditions of the problem to express the wave functions in these two regions in terms of only a single normalization constant.

(c) [20%] Calculate the probability current for the region \(x > 0\).

(d) [20%] Discuss qualitatively how the previous result would change if the energy of the eigenstate were \(E_1 > V_1\).
Problem 2 – Spin 1/2 particle

An electron has the spin state
\[
|\psi\rangle = \cos \alpha |+\rangle_z + e^{i\beta} \sin \alpha |-\rangle_z,
\]
where \( |\pm\rangle_z \) are the eigenvectors of \( \hat{S}_z \). This operator represents the component of the electron spin in the direction \( z \), and has eigenvalues \( s_z = \pm \frac{\hbar}{2} \), i.e. \( \hat{S}_z |\pm\rangle_z = \pm \frac{\hbar}{2} |\pm\rangle_z \). The coefficients \( \alpha \) and \( \beta \) are real constants.

The spin projections in the directions \( x \) and \( y \) are represented, respectively, by the operators \( \hat{S}_x \) and \( \hat{S}_y \). On the basis \( |\pm\rangle_z \), these operators are represented by the matrices \( \hat{S}_d \equiv \frac{\hbar}{2} \sigma_d \), where the symbol \( d = \{x, y, z\} \) stands for the spatial direction, \( x \), \( y \) or \( z \), and \( \sigma_d \) are the Pauli matrices.

(a) [10\%] Is \( \hat{S}_z \) an observable? If so, describe an experimental apparatus to measure it.

(b) [15\%] Find the conditions over \( \alpha \) and \( \beta \) so that \( |\psi\rangle \) is an eigenstate of \( \hat{S}_x \).

(c) [20\%] Write the expression for the state \( |\psi\rangle \) on the basis of eigenvectors \( |\pm\rangle_y \) of \( \hat{S}_y \).

(d) [35\%] Find the direction \( \mathbf{n} \equiv (\phi, \theta) \) in space aligned with the spin \( |\psi\rangle \) (i.e. the electron polarization), where the unit vector is \( \mathbf{n} = \sin \theta \cos \phi \mathbf{x} + \sin \theta \sin \phi \mathbf{y} + \cos \theta \mathbf{z} \) and the angles \( \phi \) and \( \theta \) represent the polar and azimuthal angles in spherical coordinates.

(e) [20\%] An experiment is designed to measure the energy difference \( \Delta E \) between the electron spin states in the presence of a constant magnetic field written as \( \vec{B} = B_y \mathbf{y} \). The interaction hamiltonian between field and electron spin is \( \hat{H}_I = -g \vec{B} \cdot \vec{S}/\hbar \), where \( g = 9.28 \times 10^{-24} \text{ J/T} \) is the dipole magnetic moment of the electron, and \( \vec{S} = \hat{S}_x \mathbf{x} + \hat{S}_y \mathbf{y} + \hat{S}_z \mathbf{z} \). If the energy difference is measured as \( \Delta E = 4.64 \times 10^{-24} \text{ J} \), find the experimental value of \( B_y \).
Problem 3 – Quantum harmonic oscillator

The number state \(|n\rangle\) (Fock state) of the one-dimensional quantum harmonic oscillator is defined as the eigenstate of the number operator \(\hat{n} = \hat{a}^{\dagger}\hat{a}\) with eigenvalue \(n\), i.e.

\[
\hat{n}|n\rangle = n|n\rangle,
\] (2)

where \(\hat{a}\) and \(\hat{a}^{\dagger}\) are the annihilation and creation operators, respectively.

(a) (10%) What is the physical meaning of \(n\)?

(b) (25%) Show that \([\hat{a}, \hat{a}^{\dagger}] = 1\).

(c) (35%) Show that only the ground state has minimum uncertainty between the observables position and momentum.

(d) (30%) The oscillator is prepared at time \(t = 0\) in the superposition state

\[
|\psi(t = 0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |2\rangle).
\] (3)

Obtain the variance of the position operator in the state \(|\psi(t)\rangle\) as a function of time \(t\) considering the free evolution of the oscillator.
Problem 4 – Detector of electrical pulses

In a semiconductor, an electron is confined by a potential that can be approximated as an infinite potential well of length $L_z$. In this structure, the effective mass of the electron is $m_{ef}$. This semiconductor can be used as a detector of electrical pulses in the following manner: a pulse can promote an electron initially in state $E_1$ (the lowest possible energy) to the next state, $E_2$, when a current is generated and then registered by an appropriate circuit.

Assume the electric pulse $E(t)$ has the form

$$F(t) = \begin{cases} F_0 \sin \left( \frac{\pi t}{\Delta t} \right), & (0 \leq t \leq \Delta t) \\ 0, & \text{any other case} \end{cases} \quad (4)$$

(a) [10%] Write (or solve the problem, if you do not recall the answer) the solutions of the eigenenergies $E_n^{(0)}$ and eigenfunctions $|\psi_n^{(0)}\rangle$ for a state $n$ of this unidimensional well.

(b) [10%] Assume that the field amplitude $F_0$ is sufficiently small, and denote by $V_p(t)$ the perturbation caused by electric field pulse. Write the time-dependent wave function of the system after the application of the pulse as a linear combination of the unperturbed solutions and the rate of change in time of the coefficient $a_2^{(1)}(t)$ associated to the first excited state of the system at a time $t > \Delta t$.

(c) [10%] Considering that $|\psi(t = 0)\rangle = |\psi_1^{(0)}\rangle$, write the general expression for the probability of finding the system in its first excited state at a time $t > \Delta t$.

(d) [25%] Choose the origin of the electrical potential as the middle of the well. Then calculate the matrix element of the interaction of the external field with the particle.

(e) [30%] Use the initial conditions of the problem to find an approximate expression for the probability that, after the pulse, the electron
is found in the first excited state.

Tip: To obtain the probability of finding the electron in the second state after the pulse is applied, integrate the coefficient $a_2^{(1)}(t)$ along the pulse duration to obtain $a_2^{(1)}(\Delta t)$.

(f) [15%] For a pulse duration of $\Delta t = 100$ fs, and a structure of GaAs with $m_{ef} = 0.07\ m_e$ and $L_z = 10$ nm, what is the minimum value of $F_0$ for this detector to have a probability of at least 1% to detect the pulse?
\begin{align*}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}}{2} \quad \hat{U}(t) = \exp \left( -\frac{i\hat{H}t}{\hbar} \right) \\
\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \hat{a}^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle \\
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) \quad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}} (\hat{a} - \hat{a}^\dagger) \quad [\hat{x}, \hat{p}] = \hbar \quad \hat{p} \equiv -i\hbar\vec{\nabla} \\
\Delta^2 \hat{x} &= \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \quad \Delta^2 \hat{x} \Delta^2 \hat{p} = \left| \frac{\langle [\hat{x}, \hat{p}] \rangle}{2i} \right|^2
\end{align*}