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JOÁS DA SILVA VENÂNCIO

THE SPINORIAL FORMALISM, WITH APPLICATIONS IN PHYSICS

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Dissertation presented to the graduation program of the Physics Department of Universidade Federal de Pernambuco as part of the duties to obtain the degree of Master of Philosophy in Physics.

Orientador:
Prof. Dr. Carlos Alberto Batista da Silva Filho
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Abstract

It is well-known that the rotation symmetries play a central role in the development of all physics. In this dissertation, the material is presented in a way which sets the scene for the introduction of spinors which are objects that provide the least-dimensional faithful representation for the group $Spin(n)$, the group that is the universal coverage of the group $SO(n)$, the group of rotations in n dimensions. With that goal in mind, much of this dissertation is devoted to studying the Clifford algebra, a special kind of algebra defined on vector spaces endowed with inner products. At the heart of the Clifford algebra lies the idea of a spinor. With these tools at our disposal, we studied the basic elements of differential geometry which enabled us to emphasise the more geometrical origin of spinors. In particular, we construct the spinor bundle which immediately lead to the notion of a spinor field which represents spin 1/2 particles, such as protons, electrons, and neutrons. A higher-dimensional generalization of the so-called monogenic multivector functions is also investigated. In particular, we solved the monogenic equations for spinor fields on conformally flat spaces in arbitrary dimension. Particularly, the massless Dirac field is a type of monogenic. Finally, the spinorial formalism is used to show that the Dirac equation minimally coupled to an electromagnetic field is separable in spaces that are the direct product of bidimensional spaces. In particular, we applied these results on the background of black holes whose horizons have topology $\mathbb{R} \times S^2 \times \ldots \times S^2$.

Keywords: Clifford algebra. Spinors. Monogenic. Dirac Equation. Separability.

Resumo

É bem conhecido que as simetrias de rotação desempenham um papel central no desenvolvimento de toda a física. Nesta dissertação, apresentamos o conteúdo de forma a estabelecer o cenário para a introdução dos chamados spinors, os quais são objetos que fornecem as representações fiéis de menor dimensão para o grupo $Spin(n)$, o grupo que é a cobertura universal do grupo $SO(n)$, o grupo das rotações em n dimensões. Para este fim, grande parte desta dissertação é dedicada ao estudo da álgebra de Clifford, um tipo especial de álgebra definida em espaços munidos de um produto interno. No coração da álgebra de Clifford está precisa definição de um espinor. Com estas ferramentas à nossa disposição, estudamos os elementos básicos de geometria diferêncial, o que nos permitiu entender sobre a origem mais geométrica de espinores. Em particular, construimos o fibrado espinorial, o qual conduziu imediatamente a noção de um campo espinorial que, por sua vez, representa com precição as partículas com spin 1/2 tais como: prótons, elétrons e neutrons. Uma generalização para dimensões mais altas do conceito de multivetores monogênicos também é investigada. Em particular, resolvemos a equação dos monogênicos para campos espinoriais em espaços conformemente planos em dimensão arbitrária. Particularmente, o campo de Dirac sem massa é um tipo de monogênico. Finalmente, o formalismo espinorial foi usado para mostrar que a equação de Dirac com massa minimamente acoplada ao campo eletromagnético é separável em espaços que são produtos diretos de espaços bidimensionais. Em particular, aplicamos estes resultados a buracos negros com horizontes topológicos $\mathbb{R} \times S^2 \times \ldots \times S^2$.

Palavras chaves: Álgebra de Clifford. Espinores. Monogenicos. Equação de Dirac. Separabilidade.

List of Symbols

$\mathcal{V}$	Vector space over a field $\mathbb{F}$: Page 11.
$\mathcal{V}^*$	Dual vector space of $\mathcal{V}$: Page 11.
$\mathbb{F}$	Field: Page 11.
$\mathbb{R}$	Field of the real numbers: Page 11.
$\mathbb{C}$	Field of the complex numbers: Page 11.
∂_μ	Differential operators $\dfrac{\partial}{\partial x^\mu}$: Page 49.
$T_{[a_1 a_2 \ldots a_p]}$	Skew-symmetric part of the tensor $T_{a_1 a_2 \ldots a_p}$: Page 12.
$\wedge_p \mathcal{V}$	Space of p-vectors: Page 12.
$\wedge \mathcal{V}$	Space of multivectors: Page 12.
$\langle\ \rangle_p, \langle \mathcal{A} \rangle_p$	Set of projection operators: Page 13.
$\wedge, \mathcal{A} \wedge \mathcal{B}$	Exterior product: Page 13.
$<\ ,\ >, < \mathcal{A}, \mathcal{B} >$	Non-degenerate inner product: Page 14.
$\mathcal{C}\ell(\mathcal{V})$	Clifford algebra of $\mathcal{V}$: Page 14.
$\lrcorner, \mathcal{A} \lrcorner \mathcal{B}$	Left contraction: Page 15.
$\widehat{\ }, \widehat{\mathcal{A}}$	Degree involution: Page 16.
$\widetilde{\ }, \widetilde{\mathcal{A}}$	Reversion: Page 17.
$^-, \overline{\mathcal{A}}$	Clifford conjugation: Page 18.
$\mathcal{M}(2, \mathbb{F})$	Algebra of 2×2 matrices over $\mathbb{F}$: Page 21.
$\star, \star\mathcal{A}$	Hodge dual: Page 23.
$\hat{\mathcal{S}}$	Reflection operator: Page 27.
$\hat{\mathcal{R}}$	Rotation operator: Page 28.
$\mathbb{D}$	Division algebra: Page 38.
$(\ ,\ ), (\boldsymbol{\psi}, \boldsymbol{\phi})$	Inner product between spinors: Page 38.
$\boldsymbol{g}$	The metric of the manifold: Page 50.
$\boldsymbol{\omega}^a{}_b\ ,\ \omega_{ab}{}^c\ ,\ \omega_{abc}$	Connection 1-form and its components: Page 57.
$\mathcal{T}^a$	Torsion 2-form: Page 57.
$\mathcal{R}^a{}_b$	Curvature 2-form: Page 57.
$\pounds_{\boldsymbol{K}}$	Lie derivative along of $\boldsymbol{K}$: Page 60.
$\Gamma(SM)$	Space of the local sections of spinorial bundle: Pages 74.
$\Gamma(\mathcal{C}M)$	Space of the local sections of Clifford bundle: Pages 82.

Contents

1. Introduction

The so-called rotational symmetries play a fundamental role in the development of physics. For instance, in classical mechanics and classical field theory the invariance by rotations gives rise to conserved quantities that allow the analytical integration of the equations of motion, whereas in quantum mechanics the irreducible representations of the rotation group are used to label quantum states. Moreover, and foremost, the Lorentz group can be seen as the group of rotations in a space with a metric of Lorentzian signature. It is worth recalling that the invariance under Lorentz transformation is the foundation of the standard particle model, the most successful physical theory of the second half of the twentieth century. In this dissertation, the material is presented in a way which sets the scene for the introduction of spinors which are objects that provide the least-dimensional faithful representations for the group $Spin(n)$, the universal coverage of the group $SO(n)$, the group of rotations in n dimensions. To this end, we introduce a special kind of algebra defined by vector spaces endowed with inner products: the Clifford algebra, also known as geometric algebra. This Algebra was created by the English mathematician William Kingdon Clifford around 1880 building on the earlier work of Hamilton on quaternions and Grassmann about exterior algebra. Although the definition of a spinor lies at the heart of the Clifford algebra, the spinors were discovered in 1913 by Élie Cartan as objects related to linear representations of simple groups; they provides a linear representation of the $SO(n)$ group. Hence, spinors are of fundamental importance in several branches of physics and mathematics and this dissertation sheds light on the role played by spinors on physics and mathematics. This dissertation splits in three chapters. In chapter 1 we introduce the Clifford algebras intimately related to the orthogonal transformations, the rotations. These algebras play a central role in the construction of the groups $Pin(n)$ and $Spin(n)$, which are the universal covering groups of the orthogonal groups $O(n)$ and $SO(n)$ respectively. Finally, at the end of chapter, we define the so-called spinors. In chapter 2 we study the basic elements of differential geometry, such as the curvature tensor, the Killing vectors and the fiber bundles which enabled us to emphasize on more geometrical origin of these objects. In particular, we construct the spinor bundle which immediately lead to the notion of a spinor field which represents spin 1/2 particles, such as protons, electrons, and neutrons. Finally, in chapter 3 the monogenic functions which are multivector functions that are annihilated from the left by the Dirac operator are reviewed and a higher-dimensional generalization these multivetor functions is also investigated. In particular, we solved the monogenic equations for spinor fields on conformally flat space in arbitrary dimension. Moreover, in chapter 3 it is present the main results of this dissertation. It is shown that the Dirac Equation coupled to a gauge field can be decoupled in even-dimensional manifolds that are the direct product of bidimensional spaces.

2. Clifford Algebra and Spinors

A vector space $\mathcal{V}$ over a field $\mathbb{F}$ is a set of vectors with an operation of addition and a rule of scalar multiplication, which assigns a vector to the product of a vector with an element of the field. Elements in $\mathbb{F}$ will be called scalars. An algebra is a vector space in which an associative multiplication between the vectors is defined. In this chapter we introduce the Clifford algebra, a special kind of algebra defined on vector spaces endowed with inner products. These algebras are intimately related to the orthogonal transformations, namely the rotations. This special algebra plays a central role in the construction of the groups $Pin(\mathcal{V})$ and $Spin(\mathcal{V})$, which are the universal covering groups of the orthogonal groups $O(\mathcal{V})$ and $SO(\mathcal{V})$ respectively, and also define the so-called spinors, which are elements of the a vector space on which a fundamental representation of these groups act.

2.1 The Exterior Algebra

Let $\mathcal{V}$ be an n-dimensional vector space over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\{\boldsymbol{e}_a\}$ an arbitrary basis for $\mathcal{V}$, where $a \in \{1, 2, ..., n\}$. We can expand any vector $\boldsymbol{V} \in \mathcal{V}$ in this basis as

$$\boldsymbol{V} = V^a\, \boldsymbol{e}_a\,, \tag{2.1}$$

where we have started to employ the Einstein summation convention on which repeated indices are summed. Associated to $\mathcal{V}$ is the dual space of $\mathcal{V}$, denoted by $\mathcal{V}^*$ whose elements are linear functionals, also called co-vectors,

$$\begin{array}{rccc} \boldsymbol{\omega} : & \mathcal{V} & \rightarrow & \mathbb{F} \\ & \boldsymbol{V} & \mapsto & \boldsymbol{\omega}(\boldsymbol{V}) \end{array} \quad , \tag{2.2}$$

which obey the rule of linearity $\boldsymbol{\omega}(\lambda\, \boldsymbol{V} + \delta\, \boldsymbol{U}) = \lambda\, \boldsymbol{\omega}(\boldsymbol{V}) + \delta\, \boldsymbol{\omega}(\boldsymbol{U})\ \forall\ \ \lambda\,, \delta \in \mathbb{F}$. If the addition of co-vectors and their multiplication by an element of the field is trivially defined it is straightforward to prove that the dual of $\mathcal{V}$ is also a vector space.

We can define the co-vectors $\boldsymbol{e}^b \in \mathcal{V}^*$ $(b = 1, 2, ..., n)$ by their action on elements of $\mathcal{V}$ as:

$$\boldsymbol{e}^a(\boldsymbol{e}_b) = \delta^a{}_b = \begin{cases} 1\,, & \text{if } a = b \\ 0\,, & \text{if } a \neq b \end{cases}\,, \tag{2.3}$$

hence the co-vectors $\{\boldsymbol{e}^a\}$ provide a basis for $\mathcal{V}^*$. So, any co-vector $\boldsymbol{\omega}$ can be expanded in this basis and written as:

$$\boldsymbol{\omega} \;=\; \omega_a\, \boldsymbol{e}^a\,, \tag{2.4}$$

where $\omega_a \;=\; \boldsymbol{\omega}(\boldsymbol{e}_a)$. It follows that if $\boldsymbol{\omega} \in \mathcal{V}^*$ is co-vector and $\boldsymbol{V} \in \mathcal{V}$ is a vector then $\boldsymbol{\omega}(\boldsymbol{V}) = \omega_a V^a \in \mathbb{F}$. We also may regard elements of $\mathcal{V}$ as linear functions on its dual $\mathcal{V}^*$ by defining $\boldsymbol{V}(\boldsymbol{\omega}) \equiv \boldsymbol{\omega}(\boldsymbol{V})$, since $\mathcal{V}^{**} \sim \mathcal{V}$.

The tensor product refers to way of constructing a big vector space out of two or more smaller vector spaces. For example, the tensor product of $\boldsymbol{V} \in \mathcal{V}$ with $\boldsymbol{\omega} \in \mathcal{V}^*$, denoted as $\boldsymbol{V} \otimes \boldsymbol{\omega}$, gives rise a new tensor $\boldsymbol{t}$ belonging to a vector space $\mathcal{V} \otimes \mathcal{V}^*$ on which the n^2 elements of the form $\boldsymbol{e}_a \otimes \boldsymbol{e}^b$ provide a basis, so that the most general element of this space can be written as $\boldsymbol{t} \;=\; t^a{}_b\, \boldsymbol{e}_a \otimes \boldsymbol{e}^b$, where $t^a{}_b = \boldsymbol{t}(\boldsymbol{e}^a, \boldsymbol{e}_b) \in \mathbb{F}$. Likewise, an arbitrary tensor $\boldsymbol{T}$ has the following abstract representation:

$$\boldsymbol{T} \;=\; T^{a_1 \ldots a_p}{}_{b_1 \ldots\, b_q}\, \boldsymbol{e}_{a_1} \otimes \ldots \otimes \boldsymbol{e}_{a_p} \otimes \boldsymbol{e}^{b_1} \otimes \ldots \otimes \boldsymbol{e}^{b_q}\,, \tag{2.5}$$

where $T^{a_1 \ldots a_p}{}_{b_1 \ldots\, b_q} \equiv \boldsymbol{T}(\boldsymbol{e}^{a_1}, \ldots, \boldsymbol{e}^{a_p}, \boldsymbol{e}_{b_1}, \ldots, \boldsymbol{e}_{b_q})$. The space of such tensors is denoted $T^p{}_q(\mathcal{V})$.

The so-called p-vectors are totally skew-symmetric tensors of degree p and the vector space generated by all them, namely $A^{a_1 \ldots a_p} = A^{[a_1 \ldots a_p]}$, is denoted by $\wedge_p \mathcal{V}$, where we identify the field $\mathbb{F}$ with $\wedge_0 \mathcal{V}$ and $\mathcal{V}$ with $\wedge_1 \mathcal{V}$. We must assume that all the indices inside square brackets take different values to ensure that the p-vector $\boldsymbol{A} \in \wedge_p \mathcal{V}$ given by

$$\boldsymbol{A} \;=\; A^{[a_1 \ldots a_p]}\, \boldsymbol{e}_{a_1} \otimes \ldots \otimes \boldsymbol{e}_{a_p}\,, \tag{2.6}$$

be non-null. Because the antisymmetry, it is a simple matter to prove that the dimension of $\wedge_p \mathcal{V}$ is:

$$\dim(\wedge_p \mathcal{V}) \;=\; \binom{n}{p} = \begin{cases} \frac{n!}{p!(n-p)!} & \text{if} \quad 0 \leq p \leq n \\ 0 & \text{if} \quad p > n \end{cases}\,, \tag{2.7}$$

from which we can note that p-vectors of degree greater than n are zero. By taking the direct sum of the spaces $\wedge_p \mathcal{V}$, we obtain the 2^n-dimensional multivector space, that is

$$\wedge \mathcal{V} \;=\; \bigoplus_{p=0}^{n} \wedge_p \mathcal{V}\,. \tag{2.8}$$

Elements in $\wedge\mathcal{V}$ are called multivectors and we denote them by $\mathcal{A}$. Because of decomposition of $\wedge\mathcal{V}$ in p-vector subspaces, we can define a set of projection operators, denoted by $\langle\ \rangle_p$, whose action is:

$$\begin{array}{rcll} \langle\ \rangle_p : \ \wedge\mathcal{V} & \rightarrow & \wedge_p\mathcal{V} & \\ \mathcal{A} & \mapsto & \langle\mathcal{A}\rangle_p & = \mathcal{A}_p\,. \end{array} \tag{2.9}$$

Notice that $\langle\ \rangle_p$ is in fact a projector, since that it maps an arbitrary multivector to its p-vector component each of the form (2.6). Thus, an arbitrary multivector $\mathcal{A}$ can be decomposed into a sum of pure degree terms

$$\begin{aligned} \mathcal{A} &= \langle\mathcal{A}\rangle_0 + \langle\mathcal{A}\rangle_1 + \ldots + \langle\mathcal{A}\rangle_n \\ &= \mathcal{A}_0 + \mathcal{A}_1 + \ldots + \mathcal{A}_n = \sum_{p=0}^{n} \mathcal{A}_p\,. \end{aligned} \tag{2.10}$$

Multivectors containing terms of only one degree are called homogeneous.

An interesting feature of the space of multivectors is that it carries naturally a product, denoted by $\wedge$, which maps two tensors of degrees p and q to a totally antisymmetric tensor of degree $(p+q)$, this is the so-called **exterior product**. Therefore, if $\boldsymbol{A}$ is a p-vector and $\boldsymbol{B}$ is a q-vector, the exterior product $\wedge$ is a map such that:

$$\begin{array}{rcl} \wedge\ :\ \wedge_p\mathcal{V}\times\wedge_q\mathcal{V} & \rightarrow & \wedge_{p+q}\mathcal{V} \\ \boldsymbol{A}\,,\boldsymbol{B} & \mapsto & \boldsymbol{A}\wedge\boldsymbol{B} \end{array}, \tag{2.11}$$

where $\boldsymbol{A}\wedge\boldsymbol{B} = (-1)^{pq}\,\boldsymbol{B}\wedge\boldsymbol{A}$ is defined as:

$$\boldsymbol{A}\wedge\boldsymbol{B} = \frac{(p+q)!}{p!\,q!}A^{[a_1\ldots a_p}B^{b_1\ldots b_q]}\boldsymbol{e}_{a_1}\otimes\ldots\otimes\boldsymbol{e}_{a_p}\otimes\boldsymbol{e}_{b_1}\otimes\ldots\otimes\boldsymbol{e}_{b_q}\,. \tag{2.12}$$

In n dimensions the set $\{1,\boldsymbol{e}_{a_1},\boldsymbol{e}_{a_1}\wedge\boldsymbol{e}_{a_2},\boldsymbol{e}_{a_1}\wedge\boldsymbol{e}_{a_2}\wedge\boldsymbol{e}_{a_3},\ldots,\boldsymbol{e}_{a_1}\wedge\ldots\wedge\boldsymbol{e}_{a_n}\}$, which contains 2^n elements furnish a basis for the space of multivectors $\wedge\mathcal{V}$. Therefore, a multivector $\mathcal{A}\in\wedge\mathcal{V}$ may be written in this basis as

$$\mathcal{A} = \underbrace{A}_{0-vector} + \underbrace{A^a\boldsymbol{e}_a}_{1-vector} + \underbrace{A^{ab}\,\boldsymbol{e}_a\wedge\boldsymbol{e}_b}_{2-vector} + \underbrace{A^{abc}\,\boldsymbol{e}_a\wedge\boldsymbol{e}_b\wedge\boldsymbol{e}_c}_{3-vector}\ldots + \underbrace{A^{a_1\ldots a_n}\,\boldsymbol{e}_{a_1}\wedge\ldots\wedge\boldsymbol{e}_{a_n}}_{n-vector}\,. \tag{2.13}$$

For instance, in $\mathcal{V}=\mathbb{R}^3$ there exist eight elements that generate this basis. These are: one 0-vector which is denoted by 1, three 1-vectors $\boldsymbol{e}_a$, three 2-vectors $\boldsymbol{e}_a\wedge\boldsymbol{e}_b$ each of the form $\boldsymbol{e}_a\wedge\boldsymbol{e}_b = \boldsymbol{e}_a\otimes\boldsymbol{e}_b - \boldsymbol{e}_b\otimes\boldsymbol{e}_a$ for each choice of $a\neq b=1,2,3$ and a single 3-vector. In particular, note that this 3-vector is written as

$$\begin{aligned} \boldsymbol{e}_a\wedge\boldsymbol{e}_b\wedge\boldsymbol{e}_c &= \boldsymbol{e}_a\otimes\boldsymbol{e}_b\otimes\boldsymbol{e}_c + \boldsymbol{e}_b\otimes\boldsymbol{e}_c\otimes\boldsymbol{e}_a + \boldsymbol{e}_c\otimes\boldsymbol{e}_a\otimes\boldsymbol{e}_b + \\ &- \boldsymbol{e}_b\otimes\boldsymbol{e}_a\otimes\boldsymbol{e}_c - \boldsymbol{e}_c\otimes\boldsymbol{e}_b\otimes\boldsymbol{e}_a - \boldsymbol{e}_a\otimes\boldsymbol{e}_c\otimes\boldsymbol{e}_b. \end{aligned}$$

Definition 1. *The exterior algebra is an associative algebra formed from $\wedge\mathcal{V}$ and the exterior product on p-vectors.*

2.2 The Clifford Algebra

The definitions of the exterior product, and of the multivectors do not depend on any inner product. However, Clifford algebras are a generalization of exterior algebras, defined in the presence of an inner product. In what follows, we will assume an inner product in our vector space. Let $\mathcal{V}$ be a vector space endowed with a non-degenerate inner product $< , >$, this means that for $\boldsymbol{V} \in \mathcal{V}$, $< \boldsymbol{V}, \boldsymbol{U} > = 0$ for all $\boldsymbol{U} \in \mathcal{V}$ if and only if $\boldsymbol{V} = 0$. The pair $(\mathcal{V}, < , >)$ is called an orthogonal space. Then the **Clifford product**, assumed to be associative and denoted by juxtaposition, of a pair of vectors is defined to be such that its symmetric part gives the inner product:

$$\boldsymbol{V}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{V} = 2 < \boldsymbol{V}, \boldsymbol{U} > \quad \forall \quad \boldsymbol{V}, \boldsymbol{U} \in \mathcal{V}. \tag{2.14}$$

Since the base used is completely arbitrary, it is convenient to adopt $\{\boldsymbol{e}_a\} = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \ldots, \hat{\boldsymbol{e}}_n\}$ as an orthonormal basis, since in this case we have $< \hat{\boldsymbol{e}}_a, \hat{\boldsymbol{e}}_b > = \pm\delta_{ab}$ and therefore $\hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b = -\hat{\boldsymbol{e}}_b\hat{\boldsymbol{e}}_a$, if $a \neq b$. Analogously, $\hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b\hat{\boldsymbol{e}}_c$ is totally skew-symmetric if $a \neq b \neq c \neq a$. Thus, any element of $\mathcal{C}\ell(\mathcal{V}, < , >)$, the Clifford algebra of $\mathcal{V}$, can be written as a linear combination of p-vectors $\hat{\boldsymbol{e}}_{a_1} \ldots \hat{\boldsymbol{e}}_{a_p}$ with $1 \leq p \leq n$. In other words, the vector space of the Clifford algebra associated to $\mathcal{V}$ is $\wedge\mathcal{V}$, then it can be written as a direct sum:

$$\mathcal{C}\ell(\mathcal{V}) = \bigoplus_{p=0}^{n} \wedge_p \mathcal{V}, \tag{2.15}$$

hence the dimension $\dim(\mathcal{C}\ell(\mathcal{V}, < , >)) = 2^n$. This decomposition introduces a multivector structure into the Clifford algebra $\mathcal{C}\ell(\mathcal{V})$. The multivector structure is unique, that is, an arbitrary element $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ can be uniquely decomposed into a sum of p-vectors $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \ldots + \mathcal{A}_n$. In what follows, for simplicity, instead of the previous notation it will be denoted by $\mathcal{C}\ell(\mathcal{V})$ the Clifford algebra associated to $\mathcal{V}$. It follows that the set $\{\hat{1}, \hat{\boldsymbol{e}}_{a_1}, \hat{\boldsymbol{e}}_{a_1}\hat{\boldsymbol{e}}_{a_2}, \ldots, \hat{\boldsymbol{e}}_{a_1} \ldots \hat{\boldsymbol{e}}_{a_n}\}$ with the identity element $\hat{1} \in \mathbb{F}$, forms a basis for $\mathcal{C}\ell(\mathcal{V})$. Therefore, a general element $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ can be written as:

$$\mathcal{A} = \underbrace{A}_{scalar} + \underbrace{A^a\hat{\boldsymbol{e}}_a}_{vector} + \underbrace{A^{ab}\hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b}_{2-vector} + \underbrace{A^{abc}\hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b\hat{\boldsymbol{e}}_c}_{3-vector} + \ldots + \underbrace{A^{a_1 \ldots a_n}\hat{\boldsymbol{e}}_{a_1} \ldots \hat{\boldsymbol{e}}_{a_n}}_{n-vector}, \tag{2.16}$$

Now, in the same way that (2.12) was defined in terms of the tensorial product, it is natural to define the exterior product in terms of the Clifford product. So, given a permutation $\sigma : \{1, 2, \ldots, n\} \to \{\sigma(1), \sigma(2), \ldots, \sigma(2)\}$ we define the exterior product as the totally antisymmetric part of the Clifford product:

$$\boldsymbol{V}_1 \wedge \boldsymbol{V}_2 \wedge \ldots \wedge \boldsymbol{V}_p := \frac{1}{p!} \sum_{\sigma \in S_p} \epsilon(\sigma) \boldsymbol{V}_{\sigma(1)} \boldsymbol{V}_{\sigma(2)} \ldots \boldsymbol{V}_{\sigma(p)}, \tag{2.17}$$

where S_p is the symmetric group whose elements are all the permutations and $\epsilon(\sigma)$ is $+1$ on even permutations, -1 otherwise. So, for example, given two vectors $\boldsymbol{V}$ and $\boldsymbol{U}$ we find that:

$$\boldsymbol{V} \wedge \boldsymbol{U} \;=\; \frac{1}{2}(\boldsymbol{V}\boldsymbol{U} - \boldsymbol{U}\boldsymbol{V})\,. \tag{2.18}$$

The equations (2.14) and (2.18) combine to give the Clifford product of two vectors:

$$\boldsymbol{V}\boldsymbol{U} \;=< \boldsymbol{V}, \boldsymbol{U} > + \boldsymbol{V} \wedge \boldsymbol{U}\,, \tag{2.19}$$

also called the geometric product.

The decomposition of the Clifford product of two vectors into a scalar term and a 2-vector term has a natural extension to general multivectors. This may be done through the use of the projection operator in terms of which we can conveniently express the inner and exterior products. So, given $\boldsymbol{A}_p \in \wedge_p\mathcal{V}$ and $\boldsymbol{B}_q \in \wedge_q\mathcal{V}$ the exterior product is written as:

$$\boldsymbol{A}_p \wedge \boldsymbol{B}_q \;:=\; \langle \boldsymbol{A}_p \boldsymbol{B}_q \rangle_{p+q}\,,$$

and the inner product as:

$$< \boldsymbol{A}_p, \boldsymbol{B}_q > := \; \langle \boldsymbol{A}_p \boldsymbol{B}_q \rangle_{|p-q|}\,. \tag{2.20}$$

Another important operation is the left contraction, denoted by $\lrcorner$, defined as follows:

$$\boldsymbol{A}_p \lrcorner \boldsymbol{B}_q \;:=\; \begin{cases} < \boldsymbol{A}_p, \boldsymbol{B}_q > & \text{if} \quad p \leq q \\ 0 & \text{if} \quad p > q \end{cases}\,. \tag{2.21}$$

The products defined above can be extended by bilinearity for the whole algebra. Using the equations (2.18) and (2.17) the Clifford product of a vector and an arbitrary multivetor is given by:

$$\boldsymbol{V}\mathcal{A} \;=\; \boldsymbol{V} \lrcorner \mathcal{A} \;+\; \boldsymbol{V} \wedge \mathcal{A} \quad \forall \quad \boldsymbol{V} \in \mathcal{V},\, \mathcal{A} \in \mathcal{C}\ell(\mathcal{V})\,. \tag{2.22}$$

Moreover, one can prove that the contration by a vector satisfies Leibniz's rule, this means that such a contraction is a derivation of the Clifford algebra $\mathcal{C}\ell(\mathcal{V})$. Indeed, a result that is extremely useful in practice is the following:

$$\begin{aligned} \boldsymbol{V} \lrcorner (\boldsymbol{V}_1 \wedge \boldsymbol{V}_2 \wedge \ldots \wedge \boldsymbol{V}_p) \;&=\; (\boldsymbol{V} \lrcorner \boldsymbol{V}_1)\boldsymbol{V}_2 \wedge \boldsymbol{V}_3 \wedge \boldsymbol{V}_4 \wedge \ldots \wedge \boldsymbol{V}_p + \qquad (2.23) \\ &-\; (\boldsymbol{V} \lrcorner \boldsymbol{V}_2)\boldsymbol{V}_1 \wedge \boldsymbol{V}_3 \wedge \boldsymbol{V}_4 \wedge \ldots \wedge \boldsymbol{V}_p + \\ &+\; (\boldsymbol{V} \lrcorner \boldsymbol{V}_3)\boldsymbol{V}_1 \wedge \boldsymbol{V}_2 \wedge \boldsymbol{V}_4 \wedge \ldots \wedge \boldsymbol{V}_p + \ldots \\ &=\; \sum_{i=1}^{p} (-1)^{i+1} < \boldsymbol{V}, \boldsymbol{V}_i > \boldsymbol{V}_1 \wedge \ldots \\ &\quad\; \wedge \check{\boldsymbol{V}}_i \wedge \ldots \wedge \boldsymbol{V}_p, \qquad (2.24) \end{aligned}$$

where the check on $\check{\boldsymbol{V}}_i$ denotes that the term should be withdrawn from the series. For instance, when $p = 2$ we find that:

$$\boldsymbol{V} \lrcorner (\boldsymbol{U} \wedge \boldsymbol{X}) = < \boldsymbol{V}, \boldsymbol{U} > \boldsymbol{X} - < \boldsymbol{V}, \boldsymbol{X} > \boldsymbol{U} \,.$$

Using the above result, by means of the equation (2.20), that in its turn ensures $\boldsymbol{V} \lrcorner \lambda = 0$ for any scalar λ, the equation (2.22) lead us at the following formula for the Clifford product of three vectors:

$$\boldsymbol{V}\boldsymbol{U}\boldsymbol{X} = < \boldsymbol{V}, \boldsymbol{U} > \boldsymbol{X} + < \boldsymbol{U}, \boldsymbol{X} > \boldsymbol{V} - < \boldsymbol{V}, \boldsymbol{X} > \boldsymbol{U} + \boldsymbol{V} \wedge \boldsymbol{U} \wedge \boldsymbol{X} \,. \tag{2.25}$$

Note that in (2.22), as expected, multiplication by a vector raises and lowers the degree of a multivector by 1. But, in general, if $\boldsymbol{A}_p$ is a p-vector and $\boldsymbol{B}_q$ is a q-vector

$$\boldsymbol{A}_p \boldsymbol{B}_q \neq \boldsymbol{A}_p \lrcorner \boldsymbol{B}_p + \boldsymbol{A}_p \wedge \boldsymbol{B}_q,$$

where $\boldsymbol{A}_p \lrcorner \boldsymbol{B}_p \in \wedge_{p-q}(\mathcal{V})$ and $\boldsymbol{A}_p \wedge \boldsymbol{B}_p \in \wedge_{p+q}(\mathcal{V})$. Using (2.22) it is not so hard to prove that the products of two homogeneous multivector decompose as:

$$\boldsymbol{A}_p \boldsymbol{B}_q = \langle \boldsymbol{A}_p \boldsymbol{B}_q \rangle_{|p-q|} + \langle \boldsymbol{A}_p \boldsymbol{B}_q \rangle_{|p-q|+2} + \ldots + \langle \boldsymbol{A}_p \boldsymbol{B}_q \rangle_{p+q} \,. \tag{2.26}$$

Definition 2. *The Clifford algebra associated to $\mathcal{V}$, denoted by $\mathcal{C}\ell(\mathcal{V})$, consists of the vector space $\wedge\mathcal{V}$ together with the Clifford product.*

2.2.1 Involutions

It is well-known that the conjugate of the conjugate of a complex number is the complex number itself. An operation, which, when applied to itself, returns the original object is called involution. In particular, the complex conjugation is a simple example of an involution. The Clifford algebra has three involutions similar to complex conjugation. The direct sum decomposition (2.15) gives $\mathcal{C}\ell(\mathcal{V})$ the structure of a $\mathbb{Z}$-graded algebra. This induces in $\mathcal{C}\ell(\mathcal{V})$ the first involution, denoted by $\widehat{\ }$, called the **degree involution** which is a linear map whose the action is defined on homogeneous multivectors by:

$$\begin{array}{rcll} \widehat{\ } : \wedge\mathcal{V} & \rightarrow & \wedge\mathcal{V} & \\ \boldsymbol{A}_p & \mapsto & \widehat{\boldsymbol{A}_p} & = (-1)^p \boldsymbol{A}_p \,. \end{array} \tag{2.27}$$

The degree involution is an automorphism[1] such that:

$$\widehat{\mathcal{A}\mathcal{B}} = \widehat{\mathcal{A}}\widehat{\mathcal{B}} \quad \forall \quad \mathcal{A}, \mathcal{B} \in \mathcal{C}\ell(\mathcal{V}). \tag{2.28}$$

Since the map $\widehat{}$ applied to itself is identity map, $\widehat{\widehat{\boldsymbol{A}_p}} = \boldsymbol{A}_p$, the eigenvalues of $\widehat{}$ are ± 1, then the degree involution induces a $\mathbb{Z}_2$-grading on $\mathcal{C}\ell(\mathcal{V})$. Under degree involution the multivectors corresponding to eigenvalue $+1$ will be called even and the space of even multivectors will be denoted $\mathcal{C}\ell^+(\mathcal{V})$, while the multivectors corresponding to eigenvalue -1 will be called odd, and belong to the subspace $\mathcal{C}\ell^-(\mathcal{V})$. Thus, given $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ we have

$$\widehat{\mathcal{A}} = \mathcal{A}_0 - \mathcal{A}_1 + \mathcal{A}_2 - \mathcal{A}_3 + \dots . \tag{2.29}$$

In particular, we can write the inner and exterior products in terms of the degree involution. Indeed, we can prove that for an arbitrary multivector $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ the following relations are satisfied:

$$\boldsymbol{V} \lrcorner \mathcal{A} = \frac{1}{2}(\boldsymbol{V}\mathcal{A} - \widehat{\mathcal{A}}\boldsymbol{V}) \tag{2.30}$$

$$\boldsymbol{V} \wedge \mathcal{A} = \frac{1}{2}(\boldsymbol{V}\mathcal{A} + \widehat{\mathcal{A}}\boldsymbol{V}), \tag{2.31}$$

where $\boldsymbol{V} \in \mathcal{V}$. Moreover, one can prove that the contration by a vector $\boldsymbol{V} \in \mathcal{V}$ obeys the following version of of the Leibniz's rule

$$\boldsymbol{V} \lrcorner (\mathcal{A}\mathcal{B}) = \boldsymbol{V} \lrcorner \mathcal{A}\mathcal{B} + \widehat{\mathcal{A}}\boldsymbol{V} \lrcorner \mathcal{B} \quad \forall \quad \mathcal{A}, \mathcal{B} \in \wedge\mathcal{V}. \tag{2.32}$$

The second involution in Clifford algebra is the called **reversion**, denoted by $\widetilde{}$, which reverses the order of vectors in any product. For instance, for a p-vector the reverse can be formed by a series of swaps of anticommuting vectors, each resulting in a minus sign. The first vector has to swap past $p-1$ vectors, the second past $p-2$, and so on. Using this it is easy matter to see that

$$\boldsymbol{V}_1 \wedge \widetilde{\boldsymbol{V}_2 \wedge \dots} \wedge \boldsymbol{V}_p = \boldsymbol{V}_p \wedge \boldsymbol{V}_{p-1} \wedge \dots \wedge \boldsymbol{V}_2 \wedge \boldsymbol{V}_1 = (-1)^{p(p-1)/2} \boldsymbol{V}_1 \wedge \boldsymbol{V}_2 \wedge \dots \wedge \boldsymbol{V}_p \tag{2.33}$$

It follows that the reversion, on a homogeneous vector, is a linear mapping such that:

$$\begin{aligned} \widetilde{} \ : \ \wedge\mathcal{V} &\rightarrow \wedge\mathcal{V} \\ \boldsymbol{A}_p &\mapsto \widetilde{\boldsymbol{A}_p} = (-1)^{p(p-1)/2}\boldsymbol{A}_p \end{aligned} \tag{2.34}$$

From the above the degree involution is an anti-automorphism, that is,

$$\widetilde{\mathcal{A}\mathcal{B}} = \widetilde{\mathcal{B}}\widetilde{\mathcal{A}} \quad \forall \quad \mathcal{A}, \mathcal{B} \in \mathcal{C}\ell(\mathcal{V}), \tag{2.35}$$

[1] An automorphism is an an invertible mapping from a set in itself. In particular, an automorphism s of $\mathcal{V}$ is said to be orthogonal with respect to $<,>$ if s leaves $<,>$ invariant, *i.e*, $< s\boldsymbol{V}, s\boldsymbol{V} > = < \boldsymbol{V}, \boldsymbol{V} >$ $\forall \boldsymbol{V} \in \mathcal{V}$, see [14].

thus, we see that the reverse of $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ is:

$$\widetilde{\mathcal{A}} = \mathcal{A}_0 + \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 + \ldots . \tag{2.36}$$

The composition of the two previous involutions is called the **Clifford conjugation**, denoted by $\overline{\mathcal{A}} = \widehat{(\widetilde{\mathcal{A}})} = \widetilde{(\widehat{\mathcal{A}})}$, which is expressed by

$$\begin{aligned} {}^{-} \ : \ \wedge\mathcal{V} &\to \wedge\mathcal{V} \\ \boldsymbol{A}_p &\mapsto \overline{\boldsymbol{A}}_p = (-1)^{p(p+1)/2}\boldsymbol{A}_p \,. \end{aligned} \tag{2.37}$$

and it is also an anti-automorphism,

$$\overline{\mathcal{A}\mathcal{B}} = \overline{\mathcal{B}}\,\overline{\mathcal{A}} \quad \forall \quad \mathcal{A}, \mathcal{B} \in \mathcal{C}\ell(\mathcal{V}). \tag{2.38}$$

Thus,

$$\overline{\mathcal{A}} = \mathcal{A}_0 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 + \ldots . \tag{2.39}$$

The Clifford conjugation can be used to determine the inverse. Indeed, if $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$, then its inverse is given by

$$\mathcal{A}^{-1} = \frac{\overline{\mathcal{A}}}{\mathcal{A}\overline{\mathcal{A}}} \,. \tag{2.40}$$

In particular, using the Clifford conjugation of a vector $\boldsymbol{U} \in \mathcal{V}$ which is given by the following relation

$$\boldsymbol{U}^{-1} = \frac{\boldsymbol{U}}{\boldsymbol{U}^2} \,, \tag{2.41}$$

we can show that the associativity property of the Clifford product ensures that it is now possible to divide by vectors. In fact, defining $\boldsymbol{B} = \boldsymbol{V}\boldsymbol{U} \ \forall \ \boldsymbol{V}, \boldsymbol{U} \in \mathcal{V}$ we have that

$$\boldsymbol{B}\boldsymbol{U} = (\boldsymbol{V}\boldsymbol{U})\boldsymbol{U} = \boldsymbol{V}(\boldsymbol{U}\boldsymbol{U}) = \boldsymbol{V}\boldsymbol{U}^2 \,,$$

Now, from (2.41) we can recover $\boldsymbol{V}$ and we eventually arrive at the following expression:

$$\boldsymbol{V} = \boldsymbol{B}\boldsymbol{U}^{-1} \,.$$

This ability to divide by vectors gives the Clifford algebra considerable power.

The reversion can be used to extend the concept of norm. Now, we are going to define the norm of an arbitrary multivector $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ as follows:

$$|\mathcal{A}|^2 = \left\langle \widetilde{\mathcal{A}}\mathcal{A} \right\rangle_0 = \left\langle \mathcal{A}\widetilde{\mathcal{A}} \right\rangle_0 . \tag{2.42}$$

For example, if $\mathcal{V} = \mathbb{R}^3$ the most general element $\mathcal{A} \in \mathcal{C}\ell(\mathbb{R}^3)$ is given by

$$\begin{aligned} \mathcal{A} &= \underbrace{a}_{scalar} + \underbrace{a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3}_{vector} + \underbrace{a_{12}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 + a_{13}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3 + a_{23}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3}_{2-vector} + \\ &+ \underbrace{a_{123}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3}_{3-vector} \,, \end{aligned} \tag{2.43}$$

then, using (2.43) and (2.14) it is straightforward to prove that:

$$|\mathcal{A}|^2 \;=\; (a)^2 + (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_{12})^2 + (a_{13})^2 + (a_{23})^2 + (a_{123})^2 \,. \qquad (2.44)$$

Since we can split a multivector into those components that, under degree involution, are even and those that are odd, every $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ can be written as:

$$\mathcal{A} \;=\; \mathcal{A}_+ + \mathcal{A}_- \,, \qquad (2.45)$$

where $\mathcal{A}_+ \;=\; \mathcal{A}_0 + \mathcal{A}_2 + \mathcal{A}_4 + \ldots$ and $\mathcal{A}_- \;=\; \mathcal{A}_1 + \mathcal{A}_3 + \mathcal{A}_5 + \ldots$. It follows that we can write $\mathcal{C}\ell(\mathcal{V}) = \mathcal{C}\ell^+(\mathcal{V}) \oplus \mathcal{C}\ell^-(\mathcal{V})$ where

$$\mathcal{C}\ell^{\pm}(\mathcal{V}) \;=\; \{\mathcal{A} \in \mathcal{C}\ell(\mathcal{V}) \,|\, \widehat{\mathcal{A}} \;=\; \pm\mathcal{A}\,\} \,. \qquad (2.46)$$

This $\mathbb{Z}_2$-grading of the Clifford algebra ensures that elements of even degree form a subalgebra, the called even subalgebra. Indeed, note that, $\mathcal{A}\mathcal{B} \in \mathcal{C}\ell^+(\mathcal{V})$ if $\mathcal{A} \in \mathcal{C}\ell^{\pm}(\mathcal{V})$ and $\mathcal{B} \in \mathcal{C}\ell^{\pm}(\mathcal{V})$ while $\mathcal{A}\mathcal{B} \in \mathcal{C}\ell^-(\mathcal{V})$ if $\mathcal{A} \in \mathcal{C}\ell^{\pm}(\mathcal{V})$ and $\mathcal{B} \in \mathcal{C}\ell^{\mp}(\mathcal{V})$ or $\mathcal{A} \in \mathcal{C}\ell^{\mp}(\mathcal{V})$ and $\mathcal{B} \in \mathcal{C}\ell^{\pm}(\mathcal{V})$.

The center Z of an algebra A consists of all those elements z of A such that $za \;=\; az$ for all a in A. We can easily prove that the center of $\mathcal{C}\ell(\mathcal{V})$, denoted by $Z(\mathcal{C}\ell(\mathcal{V}))$, depends on the dimension of space as follows:

$$Z(\mathcal{C}\ell(\mathcal{V})) = \begin{cases} \wedge_0\mathcal{V}\,, & \text{if } n \text{ is even} \\ \wedge_0\mathcal{V} \oplus \wedge_n\mathcal{V}, & \text{if } n \text{ is odd} \end{cases} \,. \qquad (2.47)$$

Before proceeding let us make as example.

Example 1: The $\mathcal{C}\ell(\mathbb{R}^3)$ algebra.

Let us work out in the vector space $\mathcal{V} = \mathbb{R}^3$, whose Clifford algebra $\mathcal{C}\ell(\mathbb{R}^3)$ is generated by $\{1,\, \hat{\boldsymbol{e}}_1,\, \hat{\boldsymbol{e}}_2,\, \hat{\boldsymbol{e}}_3,\, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2,\, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3,\, \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3,\, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3\}$, which contains $2^3 \;=\; 8$ elements, where $\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1 \;=\; \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2 \;=\; \hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_3 \;=\; 1$ and $\hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b \;=\; -\hat{\boldsymbol{e}}_b\hat{\boldsymbol{e}}_a$ if $a \neq b$. A priori just for the sake of simplicity in notation, let us denote the product $\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \;\equiv\; \boldsymbol{I}$ and in the next section we discuss this important symbol. The reversion this symbol is $\widetilde{\boldsymbol{I}} \;=\; -\boldsymbol{I}$ and their square is equal to

$$\begin{aligned} \boldsymbol{I}^2 \;&=\; \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \;=\; -\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \;=\; \hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \;=\; -\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \\ &=\; -\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \;=\; -\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_3 \\ \Longrightarrow&\;\; \boldsymbol{I}^2 \;=\; -1 \,. \end{aligned} \qquad (2.48)$$

The symbol $\boldsymbol{I}$ commutes with all vectors in three dimensions, using this one can prove that $\boldsymbol{I}\mathcal{A} \;=\; \mathcal{A}\boldsymbol{I} \quad \forall \quad \mathcal{A} \in \mathcal{C}\ell(\mathbb{R}^3)$ given by (2.43). $\mathcal{C}\ell^+(\mathbb{R}^3)$, the the even subalgebra of $\mathcal{C}\ell(\mathbb{R}^3)$, is generated by $\{1,\, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2,\, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3,\, \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3\}$ which contains $\frac{1}{2}2^3 = 4$ elements in terms of which a multivector $\mathcal{A}_+ \in \mathcal{C}\ell^+(\mathbb{R}^3)$ can be written as:

$$\mathcal{A}_+ = \underbrace{a}_{scalar} + \underbrace{a_{12}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2}_{2-vector} + \underbrace{a_{13}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3}_{2-vector} + \underbrace{a_{23}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3}_{2-vector} \,.$$

Defining $\boldsymbol{i} = \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3$, $\boldsymbol{j} = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3$, $\boldsymbol{k} = \boldsymbol{i}\boldsymbol{j} = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2$, we find that $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, *i,e.,* is the quaternion algebra $\mathbb{H}$. Thus, the even subalgebra of $\mathcal{C}\ell(\mathbb{R}^3)$ is isomorphic to the quaternion algebra, as can be seen by the following correspondences:

$$\begin{array}{cc} \mathbb{H} & \mathcal{C}\ell^+(\mathbb{R}^3) \\ \hline 1 & 1 \\ \boldsymbol{i} & \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \\ \boldsymbol{j} & \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3 \\ \boldsymbol{k} & \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 \end{array} \tag{2.49}$$

The basis vectors $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ can also be represented by matrices $\sigma_1\,, \sigma_2$ and σ_3 given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;;\; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \;;\; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the so-called Pauli matrices, since these matrices in accordance with this representation satisfy the same algebra, namely $\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = \mathbb{I}\,, \sigma_a\sigma_b = -\sigma_b\sigma_a$ if $a \neq b$. The quaternion algebra admits the following matrix representation

$$1 \simeq \mathbb{I}\,; \quad \boldsymbol{i} = \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \simeq i\sigma_1\,; \quad \boldsymbol{j} = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3 \simeq i\sigma_2\,; \quad \boldsymbol{k} = \boldsymbol{i}\boldsymbol{j} = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 \simeq i\sigma_3\,,$$

where $\mathbb{I}$ is the 2×2 identity matrix. Equivalently, the multivetor $\mathcal{A} \in \mathcal{C}\ell(\mathbb{R}^3)$ can be represented as:

$$\begin{aligned} [\mathcal{A}] &= a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + a_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \\ &+ a_{12}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + a_{13}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + a_{23}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + a_{123}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \\ &\Longrightarrow [\mathcal{A}] = \begin{bmatrix} z_1 & z_3 \\ z_2 & z_4 \end{bmatrix}, \end{aligned} \tag{2.50}$$

where $z_1 = (a + a_3) + i(a_{12} + a_{123})$, $z_2 = (a_1 + a_{13}) + i(a_2 + a_{23})$, $z_3 = (a_1 - a_{13}) - i(a_2 - a_{23})$, and $z_4 = (a - a_3) - i(a_{12} - a_{123})$. Note also that $\mathcal{A}_+ \in \mathcal{C}\ell^+(\mathbb{R}^3)$ is represented by

$$[\mathcal{A}] = \begin{bmatrix} w_1 & -{w_2}^* \\ w_2 & {w_1}^* \end{bmatrix}, \tag{2.51}$$

where $w_1 = a + ia_{12}$, $w_2 = a_{13} + ia_{23}$ and w_i^* is the complex conjugate of w_i $(i = 1, 2)$. By equation (2.47), the center of $\mathcal{C}\ell(\mathbb{R}^3)$ is the subalgebra of scalars and 3-vectors, namely $\wedge_0\mathcal{V} \oplus \wedge_3\mathcal{V} = \{a + a_{123}\boldsymbol{I}\}$. Note that $\sigma_1\sigma_2\sigma_3 = i\mathbb{I}$ and the symbol $\boldsymbol{I}$ can therefore be viewed as the unit imaginary[2] i and the combination of a scalar and a pseudoscalar as a complex number. This implies the center of $\mathcal{C}\ell(\mathbb{R}^3)$ is isomorphic to the complex field $\mathbb{C}$, that is,

$$Z(\mathcal{C}\ell(\mathbb{R}^3)) = \wedge_0\mathcal{V} \oplus \wedge_3\mathcal{V} \simeq \mathbb{C}\,. \tag{2.52}$$

The correspondences $\hat{\boldsymbol{e}}_1 \simeq \sigma_1$, $\hat{\boldsymbol{e}}_2 \simeq \sigma_2$ and $\hat{\boldsymbol{e}}_3 \simeq \sigma_3$ establish the isomorphism $\mathcal{C}\ell(\mathbb{R}^3) \simeq \mathcal{M}(2, \mathbb{C})$[3] with the following correspondences of the basis elements:

$\mathcal{M}(2,\mathbb{C})$	$\mathcal{C}\ell(\mathbb{R}^3)$
$\mathbb{I}$	1
$\sigma_1, \sigma_2, \sigma_3$	$\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$
$\sigma_1\sigma_2, \sigma_1\sigma_3, \sigma_2\sigma_3$	$\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3, \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3$
$i\mathbb{I}$	$\boldsymbol{I}$

(2.53)

According to with what was seen above, the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^3)$ contains two subalgebras: the center $Z(\mathcal{C}\ell(\mathbb{R}^3))$ which is isomorphic to the complex field $\mathbb{C}$ and $\mathcal{C}\ell^+(\mathbb{R}^3)$ isomorphic to the quaternion algebra $\mathbb{H}$. Now, since the elements of $\mathbb{C}$ and $\mathbb{H}$ commute, $z\boldsymbol{q} = \boldsymbol{q}z$ for $z \in \mathbb{C}\,,\ \boldsymbol{q} \in \mathbb{H}$ and that, as real algebra, $\mathcal{C}\ell(\mathbb{R}^3)$ is generated by $\mathbb{C}$ and $\mathbb{H}$, noting that $(\dim\mathbb{C})(\dim\mathbb{H}) = \dim\mathcal{C}\ell(\mathbb{R}^3)$, we are left with the following conclude:

$$\mathcal{C}\ell(\mathbb{R}^3) \simeq \mathbb{C} \otimes \mathbb{H}. \tag{2.54}$$

□

2.2.2 Pseudoscalar, Duality Transformation and Hodge Dual

The object $\boldsymbol{I}$ mentioned in the previous example is an important element of $\mathcal{C}\ell(\mathcal{V})$. This is the highest degree element in a given algebra, the so-called **pseudoscalar**[4]. For a given vector space the highest degree element exists and is unique up to a multiplicative scalar. The exterior product of n vectors is therefore a multiple of the unique pseudoscalar for $\mathcal{C}\ell(\mathcal{V})$. Given an orthonormal basis $\{\boldsymbol{e}_a\}$ $(a = 1, 2, \ldots, n)$,

[2] The symbol i is an element which commutes with all others, which is not necessarily a property of $\boldsymbol{I}$. But, in this case it commutes with all elements and squares to -1. It is therefore a further candidate for a unit imaginary.

[3] $\mathcal{M}(2, \mathbb{C})$ denotes the algebra of 2×2 matrices over $\mathbb{C}$

[4] Directed volume element and volume form are alternative names for the pseudoscalar.

we shall define the pseudoscalar to be $\boldsymbol{I} = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 \ldots \hat{\boldsymbol{e}}_n$. This convention is well-defined if we do not change the orientation of the basis. Let us consider that the space $\mathcal{V} = \mathbb{R}^{p,q}$ has dimension $n = p + q$ with p vectors whose square is equal to 1 and q vectors whose square is equal to -1 and $s = |p - q|$ the signature of the inner product. Noting that the $p+1$ vector has square -1, the norm of $\boldsymbol{I}$ depends on the dimension of space and the signature in the following form:

$$\begin{aligned} |\boldsymbol{I}|^2 &= \left\langle \widetilde{\boldsymbol{I}}\boldsymbol{I} \right\rangle_0 = \langle \hat{\boldsymbol{e}}_q\hat{\boldsymbol{e}}_{q-1} \ldots \hat{\boldsymbol{e}}_{p+1}\hat{\boldsymbol{e}}_p \ldots \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 \ldots \hat{\boldsymbol{e}}_p\hat{\boldsymbol{e}}_{p+1} \ldots \hat{\boldsymbol{e}}_{q-1}\hat{\boldsymbol{e}}_q \rangle_0 \\ &= \hat{\boldsymbol{e}}_1^2\hat{\boldsymbol{e}}_2^2 \ldots \hat{\boldsymbol{e}}_p^2\hat{\boldsymbol{e}}_{p+1}^2 \ldots \hat{\boldsymbol{e}}_{q-1}^2\hat{\boldsymbol{e}}_q^2 = (-1)^q \\ \Longrightarrow & \ |\boldsymbol{I}|^2 = (-1)^{\frac{1}{2}(n-s)} . \end{aligned} \tag{2.55}$$

In particular $|\boldsymbol{I}|^2 = 1$ if the signature is Euclidean, $s = n$, and $|\boldsymbol{I}|^2 = -1$ if Lorentzian signature, $s = (n - 2)$. When a multivector $\mathcal{A}$ is homogeneous $|\mathcal{A}|^2 = \widetilde{\mathcal{A}}\mathcal{A}$. So, immediately we see that the sign of $\boldsymbol{I}^2$ is specified by:

$$\boldsymbol{I}^2 = (-1)^{\frac{1}{2}n(n-1)}\widetilde{\boldsymbol{I}}\boldsymbol{I} = (-1)^{\frac{1}{2}[n(n-1)+(n-s)]} . \tag{2.56}$$

Another property of the pseudoscalar is that it defines an orientation. Apart from a sign, the choice of $\boldsymbol{I}$ is independent of the choice of orthonormal basis. Choosing a sign amounts to choosing an orientation for $\mathcal{V}$ which is swapped by exchange of any pair of vectors. Since the dimension of λ_n is 1, given any vectors $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_n$ it follows that

$$\boldsymbol{V}_1 \wedge \boldsymbol{V}_2 \wedge \ldots \wedge \boldsymbol{V}_n = \lambda_n \boldsymbol{I}, \tag{2.57}$$

where[5] $\lambda_n \in \mathbb{F}$. Given the independent vectors $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots,$ and $\boldsymbol{V}_n$ their exterior product will either have the same sign as $\boldsymbol{I}$, or the opposite sign. Those with the same sign are said to have a positive orientation, otherwise have a negative orientation. In particular, the pseudoscalar $\boldsymbol{I} = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3$ of $\mathcal{C}\ell(\mathbb{R}^3)$ is always chosen to be positive orientation.

An important property is that the pseudoscalar commutes with all vectors in odd dimension while in even dimension it anticomutes with all vectors. Using this, it is an immediate consequence that in three dimensions $\boldsymbol{I}\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3$. Then, $\boldsymbol{I}(\hat{\boldsymbol{e}}_1 \wedge \hat{\boldsymbol{e}}_2) = \boldsymbol{I}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_2 = -\hat{\boldsymbol{e}}_3 = -\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2$ where $\times$ denote the vector cross product. It follows that for any two vectors $\boldsymbol{V}, \boldsymbol{U} \in \mathbb{R}^3$ the vector cross product is defined in Clifford algebra as:

$$\boldsymbol{V} \times \boldsymbol{U} = -\boldsymbol{I}(\boldsymbol{V} \wedge \boldsymbol{U}) . \tag{2.58}$$

Note that, in this case, the result of the product of $\boldsymbol{I} \in \wedge_3\mathcal{V}$ with the 2-vector $\wedge_2\mathcal{V}$ is a vector $\wedge_1\mathcal{V}$, but this results is not valid in any dimension. This product only exists in three dimensions. In general we have the product of the pseudoscalar

[5] One can prove that $|\lambda_n|$ is the analogous of the volume of a parallelepiped which is generated by the vectors $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_{n-1}$ and $\boldsymbol{V}_n$, see [16, 1, 4].

$\boldsymbol{I} \in \wedge_n\mathcal{V}$ with a homogeneous multivector $\boldsymbol{A}_p \in \wedge_p\mathcal{V}$ is another homogeneous multivector $\boldsymbol{I}\boldsymbol{A}_p \in \wedge_{n-p}\mathcal{V}$. This operation is called a **duality transformation**. In this language, the equation (2.58) means that the 2-vector was mapped to a vector by a duality transformation. Note that this is valid just on $n = 3$ and by this it is commom refer to the 2-vector $\boldsymbol{V} \wedge \boldsymbol{U}$ as a pseudovector in three dimensions.

All properties relative to commutation and anticomutation of $\boldsymbol{I}$ are contained in the following equation:

$$\boldsymbol{I}\boldsymbol{A}_p = (-1)^{p(n-1)}\boldsymbol{A}_p\boldsymbol{I}\,. \tag{2.59}$$

The pseudoscalar can be used to define an important operation, denoted by $\star$, called **Hodge dual**. This is a linear map from the space of p-vectors to the space of $(n-p)$-vectors:

$$\begin{array}{rcl} \star \;:\; \wedge_p\mathcal{V} & \to & \wedge_{n-p}\mathcal{V} \\ \boldsymbol{A}_p & \mapsto & \star\boldsymbol{A}_p = \widetilde{\boldsymbol{A}_p}\boldsymbol{I}\,. \end{array} \tag{2.60}$$

Note that the Hodge map depends of the choice of orientation. By linearity one can extend this definition to inhomogeneuos multivectors of the $\mathcal{C}\ell(\mathcal{V})$ [1]

$$\star\mathcal{A} = \widetilde{\mathcal{A}}\boldsymbol{I}\,, \tag{2.61}$$

and since the dimension $\dim(\wedge_p\mathcal{V}) = \binom{n}{p} = \binom{n}{n-p} = \dim(\wedge_{n-p}\mathcal{V})$ we have an isomorphism between $\wedge_p\mathcal{V}$ and $\wedge_{n-p}\mathcal{V}$. In particular, in three dimensions, using (2.61) one finds by anticommutation that:

$$\begin{array}{c} \star 1 = \boldsymbol{I} = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \quad ; \quad \star\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 \quad ; \quad \star\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_1 \quad ; \quad \star\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 \quad ; \\ \star(\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2) = \hat{\boldsymbol{e}}_3 \quad ; \quad \star(\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3) = \hat{\boldsymbol{e}}_2 \quad ; \quad \star(\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3) = \hat{\boldsymbol{e}}_1 \quad ; \quad \star\boldsymbol{I} = 1\,. \end{array} \tag{2.62}$$

By equation (2.56) it follows the (2.61) can be written as:

$$\star\mathcal{A} = (-1)^{\frac{1}{2}[n(n-1)+(n-s)]}\widetilde{\mathcal{A}\boldsymbol{I}} \tag{2.63}$$

with this form the above equation makes clear that the Hodge dual map depends on the signature of the inner product and of the choice of orientation. In particular, the dual of the dual of a p-vector $\boldsymbol{A} \in \wedge_p\mathcal{V}$ is proportional to identity

$$\star\star\boldsymbol{A} = (-1)^{[p(n-p)+\frac{1}{2}(n-s)]}\boldsymbol{A} \tag{2.64}$$

Example 2: Maxwell's equations .

The Clifford algebra treatment is useful in several branches of the physics, since it provides a more compact formulation, for example, the Eletromagnetism. The four Maxwell's equations can be united into a single equation. Let us define the multivector operator $\mathcal{D}$ by

$$\mathcal{D} = \frac{\partial}{\partial t} + \nabla, \tag{2.65}$$

where $\nabla = \hat{\boldsymbol{e}}_1 \frac{\partial}{\partial x_1} + \hat{\boldsymbol{e}}_2 \frac{\partial}{\partial x_2} + \hat{\boldsymbol{e}}_3 \frac{\partial}{\partial x_3}$ is the usual gradient. The electromagnetic field strength is represented by the multivetor $\mathcal{F}$ and the two scalar densities charge ρ and vectorial charge current $\boldsymbol{J}$ are combined into a single multivector $\mathcal{J}$ given by

$$\mathcal{F} = \boldsymbol{E} + \boldsymbol{I}\boldsymbol{B} \quad ; \quad \mathcal{J} = \rho - \boldsymbol{J}\,, \tag{2.66}$$

where the vector fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are the eletric field and the magnetic induction, respectively. Since the pseudoscalar commutes with all vectors in three dimensios, using the equations (2.19) and (2.58), the Clifford action of $\mathcal{D}$ on $\mathcal{F}$ is:

$$\begin{aligned}
\mathcal{D}\mathcal{F} &= (\frac{\partial}{\partial t} + \nabla)(\boldsymbol{E} + \boldsymbol{I}\boldsymbol{B}) \\
&= \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial (\boldsymbol{I}\boldsymbol{B})}{\partial t} + \nabla \boldsymbol{E} + \nabla(\boldsymbol{I}\boldsymbol{B}) \\
&= \frac{\partial \boldsymbol{E}}{\partial t} + (< \nabla, \boldsymbol{E} > + \nabla \wedge \boldsymbol{E}) + \boldsymbol{I}\left[\frac{\partial \boldsymbol{B}}{\partial t} + (< \nabla, \boldsymbol{B} > + \nabla \wedge \boldsymbol{B})\right] \\
&= \underbrace{< \nabla, \boldsymbol{E} >}_{scalar} + \underbrace{\left(\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B}\right)}_{vector} + \underbrace{\boldsymbol{I}\left(\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E}\right)}_{2-vector} + \\
&+ \underbrace{\boldsymbol{I} < \nabla, \boldsymbol{B} >}_{3-vector}\,.
\end{aligned} \tag{2.67}$$

From above identity, the Maxwell equations can be written as the following compact formula

$$\mathcal{D}\mathcal{F} = \mathcal{J}. \tag{2.68}$$

Indeed, by comparing the both sides we immediately find that:

$$\begin{aligned}
< \nabla, \boldsymbol{E} > &= \rho\,, \\
\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} &= \boldsymbol{J}\,, \\
\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} &= 0\,, \\
< \nabla, \boldsymbol{B} > &= 0\,.
\end{aligned} \tag{2.69}$$

□

2.2.3 Periodicity

Now it is useful to introduce the periodicity of the Clifford algebras from which we can relate any Clifford algebra to a number of low-dimensional Clifford algebras. The structure of a real Clifford algebra is determined by the dimension of the vector space and the signature of the metric. Then, we take the field $\mathbb{F}$ to be the real numbers $\mathbb{R}$ and let us denote the Clifford algebra of the vector space $\mathbb{R}^{p,q}$ by $\mathcal{C}\ell_{p,q}$, *i.e*, $\mathcal{C}\ell_{p,q} = \mathcal{C}\ell(\mathbb{R}^{p,q})$. The Clifford algebra $\mathcal{C}\ell_{0,1}$ is generated by $\{1, \hat{\boldsymbol{e}}_2\}$ where $\hat{\boldsymbol{e}}_2^2 = -1$, it is therefore isomorphic to the algebra of complex numbers

$$\mathcal{C}\ell_{0,1} \simeq \mathbb{C}, \tag{2.70}$$

but the Clifford algebra $\mathcal{C}\ell_{1,0}$ with basis $\{1, \hat{\boldsymbol{e}}_1\}$ where now $\hat{\boldsymbol{e}}_1^2 = 1$ is not related to any known algebra. We shall note that the elements $f_1 = \frac{1}{2}(1 + \hat{\boldsymbol{e}}_1)$ and $f_2 = \frac{1}{2}(1 - \hat{\boldsymbol{e}}_1)$ form a new basis $\{f_1, f_2\}$, with $f_1^2 = f_1, f_2^2 = f_2$ and $f_1 f_2 = f_2 f_1 = 0$, it follows that f_1 and f_2 each span mutually orthogonal one-dimensional subalgebras, each of which is isomorphic to the field $\mathbb{R}$. Then,

$$\mathcal{C}\ell_{1,0} \simeq \mathbb{R} \oplus \mathbb{R}\,. \tag{2.71}$$

By multiplication of the elements of $\{1, \hat{\boldsymbol{e}}_1\}$ and $\{1, \hat{\boldsymbol{e}}_2\}$ we can construct a basis for Clifford algebra $\mathcal{C}\ell_{1,1}$ for which a possibility is $\{1, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\}$. Then, the algebra $\mathcal{C}\ell_{1,1}$ can be naturally associated to algebra $\mathcal{C}\ell_{2,0}$ which, in its turn, is isomorphic to $\mathcal{M}(2, \mathbb{R})$ [2]. So,

$$\mathcal{C}\ell_{1,1} \simeq \mathcal{M}(2, \mathbb{R})\,. \tag{2.72}$$

In particular, one can easily prove that [12, 1, 2]

$$\mathcal{C}\ell_{0,2} \simeq \mathbb{H}\,.$$

Actually, these are some general features of associative algebras [1, 14].

The following theorem, called Periodicity theorem and demonstrated in [1], establish important isomorphisms between different Clifford algebras.

Theorem 1. *Let $\mathcal{C}\ell_{p,q}$ be the Clifford algebra for the vector space $\mathbb{R}^{p,q}$:*

$$\begin{aligned} \mathcal{C}\ell_{p+1,q+1} &\simeq \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell_{p,q} \quad ; \\ \mathcal{C}\ell_{q+2,p} &\simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{p,q} \quad ; \\ \mathcal{C}\ell_{q,p+2} &\simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{p,q} \quad , \end{aligned} \tag{2.73}$$

where $p > 0$ or $q > 0$.

This theorem immediately implies the following corollary:

Corollary 1. *Any Clifford algebra $\mathcal{C}\ell_{p,q}$ can be determined from the algebras $\mathcal{C}\ell_{0,1}$, $\mathcal{C}\ell_{1,0}, \mathcal{C}\ell_{1,1}, \mathcal{C}\ell_{0,2}$ and $\mathcal{C}\ell_{2,0}$.*

Clifford algebras admit a **periodicity of dimension 8** over the real numbers. Indeed, using the latter theorem, we obtain the following relations

$$\mathcal{C}\ell_{0,4} \simeq \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \quad , \quad \mathcal{C}\ell_{4,0} \simeq \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \quad , \quad \mathcal{C}\ell_{0,8} \simeq \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{0,4} \,.$$

From above identity, we are left with the following final relation

$$\mathcal{C}\ell_{p,q+8} \simeq \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{0,8} \otimes \mathcal{C}\ell_{p,q} \,. \tag{2.74}$$

thus achieving the periodicity that we were looking for.

2.3 The Spin Groups

In this section we establish the connection between the Clifford algebra and rotations, which is the most relevant application of the Clifford algebra. Indeed, this algebra provides a very clear and compact method for performing rotations, which is considerably more powerful than working with matrices. Let $\boldsymbol{n} \in \mathcal{V}$ be a non-null vector, $\boldsymbol{n}^2 = <\boldsymbol{n}, \boldsymbol{n}> \neq 0$. This elements are invertible in $\mathcal{C}\ell(\mathcal{V})$,

$$\boldsymbol{n}^{-1} = \frac{\boldsymbol{n}}{<\boldsymbol{n}, \boldsymbol{n}>} \,. \tag{2.75}$$

The set of all the non-null vectors generates a group under Clifford product, this is called **Clifford group**

$$\Gamma = \left\{ \boldsymbol{n}_p \ldots \boldsymbol{n}_2 \boldsymbol{n}_1 \in \mathcal{C}\ell(\mathcal{V}) \,|\, \boldsymbol{n}_i \text{ is non-null} \right\} \,. \tag{2.76}$$

Note that $\boldsymbol{n}^{-1} = \pm \boldsymbol{n}$ when $\boldsymbol{n}$ is a normalized vector. Any vector $\boldsymbol{V} \in \mathcal{V}$ can be decomposed as:

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{V}\boldsymbol{n}\boldsymbol{n}^{-1} \\ &= (<\boldsymbol{V}, \boldsymbol{n}> + \boldsymbol{V} \wedge \boldsymbol{n})\boldsymbol{n}^{-1} \\ &= <\boldsymbol{V}, \boldsymbol{n}> \boldsymbol{n}^{-1} + (\boldsymbol{V} \wedge \boldsymbol{n})\boldsymbol{n}^{-1} \\ \Longrightarrow & \boldsymbol{V} = \boldsymbol{V}_{\parallel} + \boldsymbol{V}_{\perp} \end{aligned} \tag{2.77}$$

where $\boldsymbol{V}_{\parallel}$ is the component of $\boldsymbol{V}$ along of $\boldsymbol{n}$ and $\boldsymbol{V}_{\perp}$ is the component of $\boldsymbol{V}$ orthogonal to $\boldsymbol{n}$ given by

$$\boldsymbol{V}_{\parallel} = <\boldsymbol{V}, \boldsymbol{n}> \boldsymbol{n}^{-1} \quad , \quad \boldsymbol{V}_{\perp} = (\boldsymbol{V} \wedge \boldsymbol{n})\boldsymbol{n}^{-1} \,. \tag{2.78}$$

One can easily verify that the above relation has the properties that we desire. In order to see this, note that the expression for $\boldsymbol{V}_\parallel$ is, in fact, the projection of $\boldsymbol{V}$ onto vector $\boldsymbol{n}$, since $<\boldsymbol{V},\boldsymbol{n}>$ is a constant and the remaining term must be the perpendicular component. Moreover, from the following inner product

$$<\boldsymbol{n},\boldsymbol{V}_\perp>=\left\langle \boldsymbol{n}(\boldsymbol{V}\wedge\boldsymbol{n})\boldsymbol{n}^{-1}\right\rangle_0=\left\langle \boldsymbol{V}\wedge\boldsymbol{n}^{-1}\right\rangle_0=0\,,$$

we check that $\boldsymbol{V}_\perp$ is perpendicular to $\boldsymbol{n}$. We can rewrite (2.77) as:

$$\begin{aligned}\boldsymbol{V}-2\boldsymbol{V}_\parallel &= \boldsymbol{V}_\perp-\boldsymbol{V}_\parallel\\ \boldsymbol{V}-2<\boldsymbol{V},\boldsymbol{n}>\boldsymbol{n}^{-1} &= (\boldsymbol{V}\wedge\boldsymbol{n})\boldsymbol{n}^{-1}-<\boldsymbol{V},\boldsymbol{n}>\boldsymbol{n}^{-1}\end{aligned}\tag{2.79}$$

If the linear operator $\hat{\mathcal{S}}:\mathcal{V}\mapsto\mathcal{V}$ is the reflection in the plane orthogonal to $\boldsymbol{n}$, we have that:

$$\begin{aligned}\hat{\mathcal{S}}(\boldsymbol{V}_\perp+\boldsymbol{V}_\parallel) &= \boldsymbol{V}_\perp-\boldsymbol{V}_\parallel\\ &= \boldsymbol{V}-2<\boldsymbol{V},\boldsymbol{n}>\boldsymbol{n}^{-1}\,,\end{aligned}\tag{2.80}$$

and by means of (2.14) and (2.18) we easily obtain that:

$$\begin{aligned}\hat{\mathcal{S}}(\boldsymbol{V}) &= (\boldsymbol{V}\wedge\boldsymbol{n}-<\boldsymbol{V},\boldsymbol{n}>)\boldsymbol{n}^{-1}\\ &= \left(\frac{\boldsymbol{VU}-\boldsymbol{UV}}{2}-\frac{\boldsymbol{VU}+\boldsymbol{UV}}{2}\right)\boldsymbol{n}^{-1}\\ &\Longrightarrow \hat{\mathcal{S}}(\boldsymbol{V})=-\boldsymbol{nVn}^{-1}\end{aligned}\tag{2.81}$$

which is valid on spaces of any dimension and $\hat{\mathcal{S}}$ is called reflection operator. It follows that for $\boldsymbol{V}\in\mathcal{V}$ the operation $\hat{\mathcal{S}}(\boldsymbol{V})\in\mathcal{V}$ and it is the reflection of the vector $\boldsymbol{V}$ with respect to the the plane orthogonal to $\boldsymbol{n}$. Note that for a normalized vector $\boldsymbol{n}^2=\pm1$ the action of $\hat{\mathcal{S}}$ with itself is $\hat{\mathcal{S}}(\hat{\mathcal{S}}(\boldsymbol{V}))=\hat{\mathcal{S}}\circ\hat{\mathcal{S}}(\boldsymbol{V})=(-1)\hat{\mathcal{S}}(\boldsymbol{nVn}^{-1})=\boldsymbol{nnVnn}=\boldsymbol{V}$, *i.e.*, $\hat{\mathcal{S}}\circ\hat{\mathcal{S}}=\mathbb{I}$. We should check that the expression for the reflection has the desired property of leaving the inner product invariant. A simple proof is given by:

$$\begin{aligned}<\hat{\mathcal{S}}(\boldsymbol{V}),\hat{\mathcal{S}}(\boldsymbol{U})> &= \frac{\hat{\mathcal{S}}(\boldsymbol{V})\hat{\mathcal{S}}(\boldsymbol{U})+\hat{\mathcal{S}}(\boldsymbol{U})\hat{\mathcal{S}}(\boldsymbol{V})}{2}=\frac{\boldsymbol{n}(\boldsymbol{VU}+\boldsymbol{UV})\boldsymbol{n}^{-1}}{2}\\ &= <\boldsymbol{V},\boldsymbol{U}>\,.\end{aligned}\tag{2.82}$$

The reflection operator is therefore a linear transformation that preserves the inner product. Moreover, it can be proved that this transformation has determinant -1. In fact, defining the action of $\hat{\mathcal{S}}$ on an arbitrary homogeneous multivector by $\hat{\mathcal{S}}(\boldsymbol{V}_1\boldsymbol{V}_2\ldots\boldsymbol{V}_p)=\hat{\mathcal{S}}(\boldsymbol{V}_1)\hat{\mathcal{S}}(\boldsymbol{V}_2)\ldots\hat{\mathcal{S}}(\boldsymbol{V}_p)$, using the following relation between the determinant and the pseudoscalar $\hat{\mathcal{S}}(\boldsymbol{I})=det(\hat{\mathcal{S}})\boldsymbol{I}$, see [4], and the equation (2.59) we find that:

$$\begin{aligned}\det(\hat{\mathcal{S}})\boldsymbol{I} &= \hat{\mathcal{S}}(\boldsymbol{I})=\hat{\mathcal{S}}(\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\ldots\hat{\boldsymbol{e}}_n)=\hat{\mathcal{S}}(\hat{\boldsymbol{e}}_1)\hat{\mathcal{S}}(\hat{\boldsymbol{e}}_2)\ldots\hat{\mathcal{S}}(\hat{\boldsymbol{e}}_n)\\ &= (-1)^n\boldsymbol{nIn}^{-1}=det(\hat{\mathcal{S}})\boldsymbol{I}=(-1)^n(-1)^{1-n}\boldsymbol{I}=-\boldsymbol{I}\\ &\Longrightarrow \det(\hat{\mathcal{S}})=-1\,.\end{aligned}\tag{2.83}$$

We say thus that a linear transformation which leaves the inner product invariant and it has determinant -1 is called **reflection**.

Now, suppose that the vector $\boldsymbol{U}$ is the reflection of the vector $\boldsymbol{V}$, $\hat{\mathcal{S}}_1(\boldsymbol{V}) = \boldsymbol{U}$, with respect to plane orthogonal to normalized vector $\boldsymbol{n}_1$ and the vector $\boldsymbol{X}$ obtained by reflection of $\boldsymbol{U}$, $\hat{\mathcal{S}}_2(\boldsymbol{U}) = \boldsymbol{X}$, with respect to plane orthogonal to $\boldsymbol{n}_2$. Thus,

$$\boldsymbol{X} = (\hat{\mathcal{S}}_2 \circ \hat{\mathcal{S}}_1)(\boldsymbol{V}) = -\boldsymbol{n}_2\boldsymbol{U}\boldsymbol{n}_2^{-1} = \boldsymbol{n}_2\boldsymbol{n}_1\boldsymbol{V}\boldsymbol{n}_1^{-1}\boldsymbol{n}_2^{-1}\,.$$

Defining $\boldsymbol{R} = \boldsymbol{n}_2\boldsymbol{n}_1$ we can now write the result of the rotation as:

$$\hat{\mathcal{R}}(\boldsymbol{V}) = \boldsymbol{R}\boldsymbol{V}\boldsymbol{R}^{-1}\,, \tag{2.84}$$

where $\hat{\mathcal{R}} = \hat{\mathcal{S}}_2 \circ \hat{\mathcal{S}}_1$ is a linear operator $\hat{\mathcal{R}} : \mathcal{V} \mapsto \mathcal{V}$ called rotation operator which represent the rotations. We shall note that in this derivation the dimension of the vector space was never specified, so that it must work in all spaces, whatever their dimension. Analogously equations (2.82) and (2.83) it is simple matter to prove that the rotation operator preserves the inner product and it has determinant 1. Although this demonstration is completely analogous to the equation (2.82), let us do it again using the projection operator. Suppose that $\hat{\mathcal{R}}(\boldsymbol{V}) = \boldsymbol{R}\boldsymbol{V}\boldsymbol{R}^{-1}$ and $\hat{\mathcal{R}}(\boldsymbol{U}) = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{-1}$, then

$$< \hat{\mathcal{R}}(\boldsymbol{V}), \hat{\mathcal{R}}(\boldsymbol{U}) >= \left\langle \boldsymbol{R}\boldsymbol{V}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{-1}\right\rangle_0 = \langle \boldsymbol{V}\boldsymbol{U}\rangle_0 =< \boldsymbol{V}, \boldsymbol{U} > \,.$$

Now, let $\hat{\mathcal{R}}_2$ be a rotation operator such that $\hat{\mathcal{R}}_1(\boldsymbol{V}) = \boldsymbol{R}_1\boldsymbol{V}\boldsymbol{R}_1^{-1} = \boldsymbol{U}$ and $\hat{\mathcal{R}}_2$ another rotation operator such that $\hat{\mathcal{R}}_2(\boldsymbol{U}) = \boldsymbol{R}_2\boldsymbol{U}\boldsymbol{R}_2^{-1} = \boldsymbol{X}$. It follows that:

$$\begin{aligned} \boldsymbol{X} &= \hat{\mathcal{R}}_2(\boldsymbol{U}) = \hat{\mathcal{R}}_2(\hat{\mathcal{R}}_1(\boldsymbol{V})) = (\hat{\mathcal{R}}_2 \circ \hat{\mathcal{R}}_2)(\boldsymbol{V}) = \hat{\mathcal{R}}(\boldsymbol{V}) \\ &= \boldsymbol{R}_2\boldsymbol{U}\boldsymbol{R}_2^{-1} = \boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{V}\boldsymbol{R}_1^{-1}\boldsymbol{R}_2^{-1} = \boldsymbol{R}\boldsymbol{V}\boldsymbol{R}^{-1}\,, \end{aligned} \tag{2.85}$$

where $\hat{\mathcal{R}} = \hat{\mathcal{R}}_2 \circ \hat{\mathcal{R}}_1$ and $\boldsymbol{R} = \boldsymbol{R}_2\boldsymbol{R}_1$. In general, for any inhomogeneous multivector $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ we have that:

$$\hat{\mathcal{R}}(\mathcal{A}) = \boldsymbol{R}\mathcal{A}\boldsymbol{R}^{-1}\,. \tag{2.86}$$

We say thus that a linear transformation which leave the inner product invariant and it has determinant 1 is called **rotation**. In particular, by what was seen above, a rotation in the plane generated by two unit vectors $\boldsymbol{n}_2$ and $\boldsymbol{n}_1$ is achieved by successive reflections with respect to the planes perpendicular $\boldsymbol{n}_2$ and $\boldsymbol{n}_1$. So, we can construct all rotations and reflections by application of an even number or an odd number of successive reflection operators. Moreover, it can be proved that in n dimensions any rotation can be decomposed as a product of at most n reflections [3].

The orthogonal group, denoted by $O(\mathcal{V})$, is the group of linear transformations on $(\mathcal{V}, <\,,\,>)$ that preserve the inner product and the special orthogonal group, denoted by $SO(\mathcal{V})$, is the subgroup of $O(\mathcal{V})$ restrict to determinant 1. The theorem below summarizes the results obtained previously

Theorem 2. *Any orthogonal transformation $T \in O(\mathcal{V})$ can be written as a composition of reflections with respect to the hyperplanes orthogonal to non-null vectors.*

The set of all normalized vectors form a group under Clifford product. Denoted by $Pin(\mathcal{V})$, this is called **pin group**

$$Pin(\mathcal{V}) = \{\boldsymbol{R} \in \mathcal{C}\ell(\mathcal{V}) | \boldsymbol{R} = \boldsymbol{n}_p \dots \boldsymbol{n}_2 \boldsymbol{n}_1, \boldsymbol{n}_i \in \mathcal{V} \text{ and } \boldsymbol{n}_i^2 = \pm 1\}. \tag{2.87}$$

Due to the $\mathbb{Z}_2$-grading, the elements of $\mathcal{C}\ell(\mathcal{V})$ split into those of even degree and those of odd degree. The even degree elements of the Pin group form a subgroup, denoted by $SPin(\mathcal{V})$, called the **spin group**

$$SPin(\mathcal{V}) = \{\boldsymbol{R} \in Pin(\mathcal{V}) | \boldsymbol{R} = \boldsymbol{n}_{2p} \dots \boldsymbol{n}_2 \boldsymbol{n}_1, \boldsymbol{n}_i \in \mathcal{V} \text{ and } \boldsymbol{n}_i^2 = \pm 1\}. \tag{2.88}$$

Note that $SPin(\mathcal{V}) = Pin(\mathcal{V}) \cap \mathcal{C}\ell^+(\mathcal{V})$. These groups are naturally defined on the Clifford algebra and can be used to implement reflections and rotations in arbitrary vectors $\boldsymbol{V} \in \mathcal{V}$. This can be seen very clear and compact below [12].

$$\begin{aligned} \boldsymbol{\mathit{Rotation + Reflection}} &: \ (-1)^p \boldsymbol{RVR}^{-1}, \ \boldsymbol{R} \in Pin(\mathcal{V}) \\ \boldsymbol{\mathit{Pure\ Rotation}} &: \ \boldsymbol{RVR}^{-1}, \ \boldsymbol{R} \in SPin(\mathcal{V}) \end{aligned} \tag{2.89}$$

where p is the degree of $\boldsymbol{R}$. Since the action of the elements of Γ on vectors is quadratic, $\boldsymbol{R}$ and $-\boldsymbol{R}$ generate the same transformation. So there is a two-to-one map between elements of $Pin(\mathcal{V})$ and rotations and reflections. Mathematically, $Pin(\mathcal{V})$ and $SPin(\mathcal{V})$ providing a double cover representation of the orthogonal groups $O(\mathcal{V})$ and $SO(\mathcal{V})$, respectively. Then we establish the following relations

$$O(\mathcal{V}) = \frac{Pin(\mathcal{V})}{\mathbb{Z}_2} \quad ; \quad SO(\mathcal{V}) = \frac{SPin(\mathcal{V})}{\mathbb{Z}_2} \tag{2.90}$$

Note that the group $Spin(\mathcal{V})$ is a subgroup of $Pin(\mathcal{V})$ formed by elements of even degree. One can also define $Pin(\mathcal{V})$ as being the group of the elements $\boldsymbol{R} \in \Gamma$ such that $\widetilde{\boldsymbol{R}}\boldsymbol{R} = \pm 1$, and $SPin(\mathcal{V})$ as being a subgroup of $Pin(\mathcal{V})$ formed by elements $\boldsymbol{R} \in \Gamma^+$ such that $\widetilde{\boldsymbol{R}}\boldsymbol{R} = \pm 1$. It follows that the group $Pin(\mathcal{V})$ has a subgroup, denoted by $Pin_+(\mathcal{V})$, whose elements are $\boldsymbol{R} \in \Gamma$ such $\widetilde{\boldsymbol{R}}\boldsymbol{R} = 1$. Analogously, the group $SPin(\mathcal{V})$ has an even subgroup, denoted by $Spin_+(\mathcal{V})$, whose elements are $\boldsymbol{R} \in \Gamma^+$ such $\widetilde{\boldsymbol{R}}\boldsymbol{R} = 1$. Other subgroups can be obtained by restriction to certain subgroups of $Pin(\mathcal{V})$ and $Spin(\mathcal{V})$. In fact, the group $\widehat{Spin}_+(\mathcal{V})$ has as its elements $\boldsymbol{R} \in \Gamma$ such that its norm defined by $|\widehat{\boldsymbol{R}}| = \langle \overline{\boldsymbol{R}}\boldsymbol{R} \rangle_0$ is equal to 1 [8, 2]. By equation (2.89), an element of the group $Spin(\mathcal{V})$ act in the vector space $\mathcal{V}$ and yields an element on the same vector space. This implies that the vector space $\mathcal{V}$ furnish a representation for the spin groups. But since the action is quadratic, it follows it is not faithful [12, 14, 1].

2.4 Spinors

In the previous section it was established that the vector space $\mathcal{V}$ provides a representation for the spins groups, since the action of elements of the spins groups in elements of $\mathcal{V}$ results in element of $\mathcal{V}$. However, since elements of the spin group which differ by a sign produce the same orthogonal transformation, it follows that this representation is not faithful. In this section we introduce the so-called spinors, that generate a vector space that provides a faithful representation for the spin group. For simplicity, we will only consider the Clifford algebra of an orthogonal space $(\mathcal{V}, <\,,\,>)$ whose dimension is even, $n = 2r$ with $r \in \mathbb{N}$.

2.4.1 Minimal Left Ideal

There is an important class of subspaces which are called **left ideals**. A left ideal $L \subset \mathcal{C}\ell(\mathcal{V})$ of an algebra $\mathcal{C}\ell(\mathcal{V})$ is a subalgebra of $\mathcal{C}\ell(\mathcal{V})$ such that:

$$\mathcal{A}\phi \in L \quad \forall \quad \phi \in L\,, \mathcal{A} \in \mathcal{C}\ell(\mathcal{V})\,, \tag{2.91}$$

i.e, L is invariant under left multiplication of the whole algebra. Since $\mathcal{C}\ell(\mathcal{V})$ has a $\mathbb{Z}_2$-grading, we can use its even subalgebra $\mathcal{C}\ell^+(\mathcal{V})$ as the representation space of $\mathcal{C}\ell(\mathcal{V})$ and define $\rho : \mathcal{C}\ell(\mathcal{V}) \mapsto \mathrm{End}(\mathcal{C}\ell^+(\mathcal{V}))$. By means of (2.27) we can split any multivector $\mathcal{A} \in \mathcal{C}\ell(\mathcal{V})$ as $\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-$, where $\mathcal{A}_\pm = \frac{1}{2}(\mathcal{A} \pm \widehat{\mathcal{A}}) \in \mathcal{C}\ell^\pm(\mathcal{V})$. We should establish in what conditions $\rho = \rho_+ + \rho_-$ such that $\rho(\mathcal{A}) = \rho_+(\mathcal{A}_+) + \rho_-(\mathcal{A}_-)$ will be a representation of $\mathcal{C}\ell(\mathcal{V})$. Note that $\mathcal{A}_+\mathcal{B} \in \mathcal{C}\ell^+(\mathcal{V})$ if $\mathcal{B} \in \mathcal{C}\ell^+(\mathcal{V})$, this implies that $\rho_+(\mathcal{A}_+)(\mathcal{B}) = \mathcal{A}_+\mathcal{B} \;\; \forall \mathcal{B} \in \mathcal{C}\ell^+(\mathcal{V})$, while $\mathcal{A}_-\mathcal{B} \in \mathcal{C}\ell^-(\mathcal{V})$ if $\mathcal{B} \in \mathcal{C}\ell^+(\mathcal{V})$, this requires that $\rho_-(\mathcal{A}_-)(\mathcal{B}) \neq \mathcal{A}_-\mathcal{B}$ when $\mathcal{B} \in \mathcal{C}\ell^+(\mathcal{V})$. If, however, we can take an odd element $\mathcal{C} \in \mathcal{C}\ell^-(\mathcal{V})$ and define

$$\rho_-(\mathcal{A}_-)(\mathcal{B}) = \mathcal{A}_-\mathcal{B}\mathcal{C} \in \forall \mathcal{B} \in \mathcal{C}\ell^+(\mathcal{V}) \tag{2.92}$$

such that $\mathcal{C}^2 = 1$, then $\rho = \rho_+ + \rho_-$ is a representation in the space $\mathcal{C}\ell^+(\mathcal{V})$. One might then wonder, how can we know if such representation is reducible? Suppose that there exists an element $\mathcal{C}_1 \in \mathcal{C}\ell^+(\mathcal{V})$ such that

$$\mathcal{C}_1^2 = 1 \quad ; \quad \mathcal{C}_1\mathcal{C} = \mathcal{C}\mathcal{C}_1\,. \tag{2.93}$$

Thus we can write $\mathcal{C}\ell^+(\mathcal{V}) = {}_+\mathcal{C}\ell^+(\mathcal{V}) \oplus {}_-\mathcal{C}\ell^+(\mathcal{V})$, where

$${}_\pm\mathcal{C}\ell^+(\mathcal{V}) = \mathcal{C}\ell^+(\mathcal{V})\frac{1}{2}(1 \pm \mathcal{C}_1)\,. \tag{2.94}$$

so that, for $\mathcal{B}_\pm \in {}_\pm\mathcal{C}\ell^+(\mathcal{V})$ we have $\mathcal{B}_\pm\mathcal{C}_1 = \pm\mathcal{B}_\pm$. Each of these spaces ${}_\pm\mathcal{C}\ell^+(\mathcal{V})$ is invariant under the action of ρ as we can see by the equation (2.93) and it is also a subalgebra of $\mathcal{C}\ell^+(\mathcal{V})$. If another element $\mathcal{C}_2 \in \mathcal{C}\ell^+(\mathcal{V})$ such that $\mathcal{C}_2^2 = 1\,, \mathcal{C}_2\mathcal{C}_1 =$

$\mathcal{C}_1\mathcal{C}_2$ and $\mathcal{C}_2\mathcal{C} = \mathcal{C}\mathcal{C}_2$ can be found, then the subspaces ${}_\pm\mathcal{C}\ell^+(\mathcal{V})$ carry no irreducible representation. So, we define another four subspaces

$$_\pm\mathcal{C}\ell^+(\mathcal{V}) = {}_\pm\mathcal{C}\ell^+(\mathcal{V})\frac{1}{2}(1 \pm \mathcal{C}_1)(1 \pm \mathcal{C}_2), \tag{2.95}$$

each of which invariant under the action of ρ. We can proceed with this construction in an analogous fashion if there are other elements $\mathcal{C}_j$ commuting with the previous ones.

The regular representation $\rho : \mathcal{C}\ell(\mathcal{V}) \mapsto \mathrm{End}(\mathcal{C}\ell(\mathcal{V}))$ given by $\rho(\mathcal{A})\mathcal{B} = \mathcal{A}\mathcal{B}$ preserves certain vector subspaces. The **minimal left ideals** are just the invariant subspaces $S \subset \mathcal{C}\ell(\mathcal{V})$ for which the map $\rho : \mathcal{C}\ell(\mathcal{V}) \mapsto \mathrm{End}(S)$ defined by $\rho(\mathcal{A})\boldsymbol{\psi} = \mathcal{A}\boldsymbol{\psi}$ is an irreducible representation. Note that the minimal left ideal contains no left ideals apart from itself and zero, it provides the least-dimensional faithful representation of $\mathcal{C}\ell(\mathcal{V})$, the so-called spinorial representation of the Clifford algebra. The minimal left ideal S is called, as a vector space, the **spinor space**, and its elements, denoted by $\boldsymbol{\psi}$, are called **spinors**. Note also that these minimal left ideals are algebras, the subalgebras of $\mathcal{C}\ell(\mathcal{V})$. Thus, under Clifford product, the space carrying a such representation of $\mathcal{C}\ell(\mathcal{V})$ will be called spinorial algebra and the choice of a different minimal left ideal gives other equivalent representations [1, 11, 14]. We will therefore establish that the spinor representation of the Clifford algebra induces a representation of any subset by restricting to left multiplication on the ideal by elements of that set. In particular it induces a representation of the Clifford group [1]. When the dimension of $\mathcal{V}$ is $n = 2r$ one can prove that the dimension of the spinor space is 2^r. When $\mathcal{C}\ell(\mathcal{V})$ is thought of as a matrix algebra, an example of a minimal left ideal is the subalgebra of matrices with all columns but the first vanishing [2, 11, 30]. In particular, $\mathcal{C}\ell(\mathcal{V}), Pin(\mathcal{V}), SPin(\mathcal{V}), O(\mathcal{V})$ and $SO(\mathcal{V})$ can be faithfully represented by $2^r \times 2^r$ matrices. In this case spinors are represented by the column vectors on which these matrices act.

By what was seen above, the minimal left ideals are of great relevance in the study of the spinors. These can be obtained by action of the whole algebra on the so-called primitive idempotent of $\mathcal{C}\ell(\mathcal{V})$, this makes clear the fundamental importance of the primitive idempotents. An element $\boldsymbol{\xi}$ of an any algebra $\mathbb{A}$ is said to be an idempotent if $\boldsymbol{\xi}^2 = \boldsymbol{\xi}$ and $\boldsymbol{\xi} \neq 0$. Particularly, such an idempotent generates a subalgebra $\boldsymbol{\xi}\mathbb{A}\boldsymbol{\xi}$. Now, if $\mathbb{A}$ is a division algebra, then the single idempotent is the identity, since if $\boldsymbol{\xi} \neq 0$, then every $\boldsymbol{\xi}$ has an inverse $\boldsymbol{\xi}^{-1}$, it implies that $\boldsymbol{\xi}^2 = \boldsymbol{\xi} \because \boldsymbol{\xi}^{-1}\boldsymbol{\xi}^2 = \boldsymbol{\xi}^{-1}\boldsymbol{\xi} = 1_\mathcal{A} \therefore \boldsymbol{\xi} = 1_\mathcal{A}$. We can split the element $\boldsymbol{\xi}$ into a sum of other two elements $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, $\boldsymbol{\xi} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2$. Hence, $\boldsymbol{\xi}$ will be idempotent if the following relations are satisfied: $\boldsymbol{\xi}_1^2 = \boldsymbol{\xi}_1, \boldsymbol{\xi}_2^2 = \boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_1\boldsymbol{\xi}_2 = -\boldsymbol{\xi}_2\boldsymbol{\xi}_1$. If such elements satisfying the conditions $\boldsymbol{\xi}_1\boldsymbol{\xi}_2 = 0$ and $\boldsymbol{\xi} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2$ cannot be found, we call $\boldsymbol{\xi}$ a primitive idempotent. Therefore, a idempotent $\boldsymbol{\xi}$ is primitive if and only if it is the single idempotent on $\boldsymbol{\xi}\mathbb{A}\boldsymbol{\xi}$ [2, 14, 17]. Let $\boldsymbol{\xi}$ be a generic primitive idempotent in $\mathcal{C}\ell(\mathcal{V})$, then any minimal left ideal S. has the form:

$$S = \mathcal{C}\ell(\mathcal{V})\boldsymbol{\xi} = \{\mathcal{A}\boldsymbol{\xi} \,|\, \mathcal{A} \in \mathcal{C}\ell(\mathcal{V})\}\,. \tag{2.96}$$

When the dimension of $\mathcal{V}$ is even the square of $\boldsymbol{I}$ depends only on the signature of the inner product. So, equation (2.56) is reduced to:

$$\begin{aligned} \boldsymbol{I}^2 &= (-1)^{\frac{1}{2}[n(n-1)+(n-s)]} \quad \forall \quad n \\ &\Longrightarrow \boldsymbol{I}^2 = (-1)^{\frac{-s}{2}}\,, \quad n = 2r\,. \end{aligned} \tag{2.97}$$

Thus, we can express its square as $\boldsymbol{I}^2 = \varepsilon^2$, with $\varepsilon = 1$ or $\varepsilon = i$. Since $\boldsymbol{I}^2 = \pm 1$, the action of $\boldsymbol{I}$ on S gives a decomposition [12]

$$S = S^+ + S^- \quad ; \quad S^\pm = \{\boldsymbol{\psi} \in S \,|\, \boldsymbol{I}\boldsymbol{\psi} = \pm\varepsilon\boldsymbol{\psi}\}\,, \tag{2.98}$$

where $S^\pm$ are subspaces of dimension 2^{r-1} whose elements are called semi-spinors. The semi-spinors are also called Weyl spinors of positive and negative chirality. These subspaces are invariant under the action of the even subalgebra. Indeed, by means of the equation (2.59) we have that $\boldsymbol{I}\mathcal{A}_+ = \mathcal{A}_+\boldsymbol{I}$ for $\mathcal{A}_+ \in \mathcal{C}\ell^+(\mathcal{V})$ and n even. Then, due to (2.98) we have:

$$\begin{aligned} \boldsymbol{I}\mathcal{A}^+\boldsymbol{\psi} &= \mathcal{A}^+\boldsymbol{I}\boldsymbol{\psi} \\ &\Longrightarrow \boldsymbol{I}\mathcal{A}^+\boldsymbol{\psi} = \pm\varepsilon\mathcal{A}^+\boldsymbol{\psi} \quad \forall \quad \mathcal{A}^+ \in \mathcal{C}\ell^+(\mathcal{V})\,. \end{aligned} \tag{2.99}$$

This means that the spinorial representation of $\mathcal{C}\ell^+(\mathcal{V})$ on S is reducible while the spinorial representation of $\mathcal{C}\ell^+(\mathcal{V})$ on $S^\pm$ is irreducible [12, 1]. In paricular, the spinorial representation of $\mathcal{C}\ell^+(\mathcal{V})$ splits in two blocks of dimension $2^{r-1} \times 2^{r-1}$. It happens that, by equation (2.88), elements of $Spin(\mathcal{V})$ are of degree even. So, by restriction, the spinorial representation of $\mathcal{C}\ell^+(\mathcal{V})$ induce a representation of dimension 2^{r-1} for $Spin(\mathcal{V})$ [14].

Though we have restricted ourselves to even-dimensional spaces, let us clarify these concepts with an example in three dimensions following the line of the previous examples. We will assume that the dimension of S is $2^{[n/2]}$, where [ ] denotes the integer part of the number inside it.

Example 3: Spinors in the minimal left ideal of $\mathcal{C}\ell(\mathbb{R}^3)$.

Let $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_2\}$ be an orthonormal basis for vector space $\mathcal{V} = \mathbb{R}^3$ where $\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_3 = 1\,, \hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b = -\hat{\boldsymbol{e}}_b\hat{\boldsymbol{e}}_a$ if $a \neq b$. The set $\{1,\, \hat{\boldsymbol{e}}_1,\, \hat{\boldsymbol{e}}_2,\, \hat{\boldsymbol{e}}_3,\, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2,\, \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3,\, \hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3,\, \boldsymbol{I}\}$ span the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^3)$ whose the most general element is:

$$\mathcal{A} = a + a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3 + a_{12}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 + a_{13}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3 + a_{23}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3 + a_{123}\boldsymbol{I}\,,$$

where $\boldsymbol{I}$ is the pseudoscalar. In three dimensions, we must find a spinor space S of dimension $\dim(S) = 2^{[3/2]} = 2$. Now, consider the element

$$\boldsymbol{\xi}_1 = \frac{1}{2}(1 + \hat{\boldsymbol{e}}_3).$$

Note that $\boldsymbol{\xi}_1$, ${\boldsymbol{\xi}_1}^2 = \frac{1}{2}(1 + \hat{\boldsymbol{e}}_3)\frac{1}{2}(1 + \hat{\boldsymbol{e}}_3) = \boldsymbol{\xi}_1$, is a primitive idempotent. It follows that the set of the form

$$S = \mathcal{C}\ell(\mathbb{R}^3)\boldsymbol{\xi}_1 = \{\mathcal{A}\boldsymbol{\xi}_1 \,|\, \mathcal{A} \in \mathcal{C}\ell(\mathbb{R}^3)\}\,,$$

is a minimal left ideal of $\mathcal{C}\ell(\mathbb{R}^3)$. Using the Clifford algebra, one finds after some manipulations that

$$\mathcal{A}\boldsymbol{\xi}_1 = [(a + a_3) + \boldsymbol{I}(a_{12} + a_{123})]\boldsymbol{\xi}_1 + [(a_1 + a_{13}) + \boldsymbol{I}(a_2 + a_{23})]\hat{\boldsymbol{e}}_1\boldsymbol{\xi}_1\,. \quad (2.100)$$

Defining $\boldsymbol{\xi}_2 = \hat{\boldsymbol{e}}_1\boldsymbol{\xi}_1$, we obtain that:

$$\mathcal{A}\boldsymbol{\xi}_1 = [(a + a_3) + \boldsymbol{I}(a_{12} + a_{123})]\boldsymbol{\xi}_1 + [(a_1 + a_{13}) + \boldsymbol{I}(a_2 + a_{23})]\boldsymbol{\xi}_2\,. \quad (2.101)$$

In the same way, we obtain

$$\mathcal{A}\boldsymbol{\xi}_2 = [(a_1 - a_{13}) - \boldsymbol{I}(a_2 + a_{23})]\boldsymbol{\xi}_1 + [(a - a_3) - \boldsymbol{I}(a_{12} + a_{123})]\boldsymbol{\xi}_2\,. \quad (2.102)$$

Note that that the pseudoscalar $\boldsymbol{I}$ commutes with all elements and squares to -1, then it can be viewed as the unit imaginary i in $\mathcal{C}\ell(\mathbb{R}^3)$, $\boldsymbol{I}\boldsymbol{\xi}_1 = i\boldsymbol{\xi}_1$. Since the action of $\mathcal{A}$ on $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are both linear combination of the set $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$, we see that:

$$S = \{\boldsymbol{\psi} \in \mathcal{C}\ell(\mathbb{R}^3) \,|\, \boldsymbol{\psi} = \psi_1\boldsymbol{\xi}_1 + \psi_2\boldsymbol{\xi}_2 \quad \forall \quad \psi_1, \psi_2 \in \mathbb{C}\}\,, \quad (2.103)$$

where $\boldsymbol{\psi}_1 = \alpha_1 + i\beta_1$ and $\boldsymbol{\psi}_2 = \alpha_2 + i\beta_2$. This space admits no proper left ideal, so the elements $\boldsymbol{\psi} = \psi^A\boldsymbol{\xi}_A$ are called spinors and $\{\boldsymbol{\xi}_A\}$ where $A \in \{1, 2\}$ form a basis for S, the called spinor basis. It is simple to prove that S is invariant by the action of $\mathcal{C}\ell(\mathbb{R}^3)$. In order to prove this explicitly, we only need to act the elements that span $\mathcal{V}$ which are $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ on the elements of S and this implies in the following spinorial representation for the vectors of the basis:

$$\hat{\boldsymbol{e}}_1\boldsymbol{\psi} = \hat{\boldsymbol{e}}_1(\psi_1\boldsymbol{\xi}_1 + \boldsymbol{\psi}_2\boldsymbol{\xi}_2) = \psi_1\boldsymbol{\xi}_2 + \boldsymbol{\psi}_2\boldsymbol{\xi}_1 \quad \Longrightarrow \hat{\boldsymbol{e}}_1 \simeq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2.104)$$

$$\hat{\boldsymbol{e}}_2\boldsymbol{\psi} = \hat{\boldsymbol{e}}_2(\psi_1\boldsymbol{\xi}_1 + \boldsymbol{\psi}_2\boldsymbol{\xi}_2) = -i\psi_1\boldsymbol{\xi}_2 + i\,\boldsymbol{\psi}_2\boldsymbol{\xi}_1 \quad \Longrightarrow \hat{\boldsymbol{e}}_2 \simeq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (2.105)$$

$$\hat{\boldsymbol{e}}_3\boldsymbol{\psi} \;=\; \hat{\boldsymbol{e}}_3(\psi_1\boldsymbol{\xi}_1 + \boldsymbol{\psi}_2\boldsymbol{\xi}_2) = \psi_1\boldsymbol{\xi}_1 - \psi_2\boldsymbol{\xi}_2 \Longrightarrow \hat{\boldsymbol{e}}_3 \simeq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad (2.106)$$

i.e., the action of $\mathcal{V}$ on S yield elements on S. Thus, as a matrix algebra, using these three last equations we can see that the spinor can be represented by:

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \in \mathbb{C}^2 \quad \text{or} \quad \boldsymbol{\psi} = \begin{bmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{bmatrix} \in \mathcal{M}(2,\mathbb{C})\boldsymbol{\xi}_1\,, \qquad (2.107)$$

where $\boldsymbol{\xi}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. If we multiply $\boldsymbol{\psi}$ on the left by an arbitrary matrix $\in \mathcal{M}(2,\mathbb{C})$ we are left with a completely analogous relation [2]. Using these matrices we arrive on the matricial representation of the multivetor $\mathcal{A}$ obtained on Example 1.

Defining the normalized vectors $\boldsymbol{n}_1 = \hat{\boldsymbol{e}}_1\,, \boldsymbol{n}_2 = \cos\theta\,\hat{\boldsymbol{e}}_1 + \sin\theta\,\hat{\boldsymbol{e}}_2$, then we can construct the following element of the group $SPin(\mathcal{V})$: $\boldsymbol{R}_1 = \boldsymbol{n}_2\boldsymbol{n}_1 = \cos\theta - \sin\theta\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2$. It follows that an element $\boldsymbol{R}_1$ of $Spin(\mathbb{R}^3)$ acts on the elements of $\mathbb{R}^3$ as $\boldsymbol{R}_1\,\hat{\boldsymbol{e}}_a\boldsymbol{R}_1^{-1}\,(a = 1,2,3)$. Thus,

$$\begin{aligned} \hat{\boldsymbol{e}}_1 \mapsto \hat{\mathcal{R}}_1(\hat{\boldsymbol{e}}_1) &= \boldsymbol{n}_2\boldsymbol{n}_1\hat{\boldsymbol{e}}_1\boldsymbol{n}_1\boldsymbol{n}_2 = \cos(2\theta)\hat{\boldsymbol{e}}_1 + \sin(2\theta)\hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_2 \mapsto \hat{\mathcal{R}}_1(\hat{\boldsymbol{e}}_2) &= \boldsymbol{n}_2\boldsymbol{n}_1\hat{\boldsymbol{e}}_2\boldsymbol{n}_1\boldsymbol{n}_2 = -\sin(2\theta)\hat{\boldsymbol{e}}_1 + \cos(2\theta)\hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \mapsto \hat{\mathcal{R}}_1(\hat{\boldsymbol{e}}_3) &= \boldsymbol{n}_2\boldsymbol{n}_1\hat{\boldsymbol{e}}_3\boldsymbol{n}_1\boldsymbol{n}_2 = \hat{\boldsymbol{e}}_3\,. \end{aligned} \qquad (2.108)$$

The result of the action of $\hat{\mathcal{R}}_1$ is therefore a rotation of 2θ on the plane generated by the two unit vectors $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$. Note that from the expression of $\boldsymbol{R}_1$ it can be represented by $\boldsymbol{R}_1 = e^{-\theta\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2}$. But we can implement rotation in any one of the $\hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b$ planes. So, we can perform others rotation by $\boldsymbol{R} = e^{-\theta\hat{\mathbf{e}}_a\hat{\mathbf{e}}_b}$ whose action yields a rotation of 2θ on the $\hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b$ plane. In particular, taking $\theta = \pi$, we have a rotation through 2θ on the $\hat{\boldsymbol{e}}_a\hat{\boldsymbol{e}}_b$ plane. In this case, the vectors remain unchanged while the spinors change of sign.

□

2.4.2 Pure Spinors

From the geometrical point of view, there exists an important class of spinors which are associated with maximal isotropic vector subspaces, observed by Cartan [3], these are the so-called **pure spinors**. For example, let $\boldsymbol{V}$ be a null vector, namely $<\boldsymbol{V},\boldsymbol{V}>= 0$. In three dimensions we can write the complex components (V_1, V_2, V_3) of this vector in terms of two elements ψ_1, ψ_2 as:

$$\begin{aligned} V_1 &= {\psi_1}^2 - {\psi_2}^2 \\ V_2 &= i({\psi_1}^2 + {\psi_2}^2) \\ V_3 &= -2\psi_1\psi_2\,, \end{aligned} \qquad (2.109)$$

which automatically guarantees the isotropic character of the vector. Indeed, the latter equation immediately implies that $V_1^2 + V_2^2 + V_3^2 = 0$. Note that these equations are solved for

$$\psi_1^2 = \frac{V_1 - iV_2}{2} \quad ; \quad \psi_2^2 = -\frac{V_1 + iV_2}{2}\,. \tag{2.110}$$

So, using the above equation and that $V_3 = -2\psi_1\psi_2$ we obtain that

$$\begin{cases} V_3\psi_1 + (V_1 - iV_2)\psi_2 = 0 \\ (V_1 + iV_2)\psi_1 - V_3\psi_2 = 0 \end{cases} \Longrightarrow \begin{bmatrix} V_3 & V_1 - iV_2 \\ V_1 + iV_2 & -V_3 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_1 \end{bmatrix} = 0\ , \tag{2.111}$$

which is the matrix representation of $\boldsymbol{V\psi} = 0$, with

$$\begin{bmatrix} V_3 & V_1 - iV_2 \\ V_1 + iV_2 & -V_3 \end{bmatrix} \begin{bmatrix} V_3 & V_1 - iV_2 \\ V_1 + iV_2 & -V_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} < \boldsymbol{V}, \boldsymbol{V} >= 0$$

and $< \boldsymbol{V}, \boldsymbol{V} >= 0$. Note that the rotation (2.108) on the vector $\boldsymbol{V}$ implies,

$$\begin{aligned} V_1' &= V_1\cos(2\theta) - V_2\sin(2\theta) \\ V_2' &= V_1\sin(2\theta) + V_2\cos(2\theta) \\ V_3' &= V_3\,. \end{aligned} \tag{2.112}$$

Using this, it is simple matter to prove that:

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \begin{bmatrix} e^{-i\theta}\psi_1 \\ e^{i\theta}\psi_2 \end{bmatrix}. \tag{2.113}$$

Note that taking $\theta = \pi$ the vector remains unchanged by the action of the rotation operator, while the elements ψ_1, ψ_2 change the sign. Thus, the pair ψ_1, ψ_2 constitutes the components of a spinor $\boldsymbol{\psi}$ which is associated a null subspace spanned by the vectors that annihilate it, $\boldsymbol{V\psi} = 0$, whose representation is given by (2.111). The spinor $\boldsymbol{\psi}$ is an example of a pure spinor.

Now, let us make a rigorous definition of a pure spinor. Formally, given a spinor $\boldsymbol{\psi} \in S$ one can construct a subspace of $\mathcal{V}$, denoted by $N_{\boldsymbol{\psi}}$, defined by:

$$N_{\boldsymbol{\psi}} = \{\boldsymbol{V} \in \mathcal{V} \,|\, \boldsymbol{V\psi} = 0\}\,. \tag{2.114}$$

Note that if $\boldsymbol{\psi}$ in non-null then for all $\boldsymbol{V}, \boldsymbol{U} \in N_{\boldsymbol{\psi}}$ we have

$$\begin{aligned} 2 < \boldsymbol{V}, \boldsymbol{U} > \boldsymbol{\psi} &= (\boldsymbol{VU} + \boldsymbol{UV})\boldsymbol{\psi} = 0 \\ &\Longrightarrow\ < \boldsymbol{V}, \boldsymbol{U} >= 0\,. \end{aligned} \tag{2.115}$$

This subspace is clearly totally null, which means that all vectors belonging to it are orthogonal to each other including to itself.

In what follows, we will only consider the complexified Clifford algebra of an even dimensional orthogonal space. The complexification of a real orthogonal space $(\mathcal{V}, <\,,\,>)$ is the space $(\mathcal{V}_{\mathbb{C}}, <\,,\,>_{\mathbb{C}})$ whose elements are of the form $\boldsymbol{V} + i\boldsymbol{U}$ for some $\boldsymbol{V}, \boldsymbol{U} \in \mathcal{V}$ and i the unit imaginary. We can define the sum and multiplication by a complex scalar $(\lambda + i\delta)$ as:

$$\begin{aligned} (\boldsymbol{V}_1 + i\boldsymbol{U}_1) + (\boldsymbol{V}_2 + i\boldsymbol{U}_2) &= (\boldsymbol{V}_1 + \boldsymbol{V}_2) + i(\boldsymbol{U}_1 + \boldsymbol{U}_2) \\ (\lambda + i\delta)(\boldsymbol{V} + i\boldsymbol{U}) &= (\lambda\boldsymbol{V} - \delta\boldsymbol{U}) + i(\lambda\boldsymbol{V} + \delta\boldsymbol{U})\,. \end{aligned} \tag{2.116}$$

The complex conjugate of $\boldsymbol{V} = \boldsymbol{V}_1 + i\boldsymbol{U}_1 \in \mathcal{V}_{\mathbb{C}}$ for $\boldsymbol{V}_1, \boldsymbol{U}_1 \in \mathcal{V}$ is defined by $\boldsymbol{V}^* = \boldsymbol{V}_1 - i\boldsymbol{U}_1$. Since $\mathcal{V}_{\mathbb{C}}$ can be obtained by the complexification $\mathcal{V}$, $\mathcal{V}_{\mathbb{C}} = \mathbb{C}\otimes\mathcal{V}$, we see that the complex dimension of $\mathcal{V}_{\mathbb{C}}$ is n and the real dimension $2n$. The inner product $<\,,\,>$ is extended to $<\,,\,>_{\mathbb{C}}$ on $\mathcal{V}_{\mathbb{C}}$ by assuming bilinearity of the inner product in the complex field. For simplicity, from now on we shall assume that the complex Clifford algebra associated to $\mathcal{V}_{\mathbb{C}}$ endowed is simply $\mathcal{C}\ell(\mathcal{V}_{\mathbb{C}})$ and the complexification of $\mathcal{C}\ell(\mathcal{V}, <\,,\,>)$ is $\mathcal{C}\ell_{\mathbb{C}}(\mathcal{V})$. From this, we say that theses algebras are isomorphic, namely:

$$\mathcal{C}\ell(\mathcal{V}_{\mathbb{C}}) \simeq \mathcal{C}\ell_{\mathbb{C}}(\mathcal{V})\,, \tag{2.117}$$

where $\mathcal{C}\ell_{\mathbb{C}}(\mathcal{V}) = \mathbb{C}\otimes\mathcal{C}\ell(\mathcal{V})$.

We say that a spinor is pure if the null subspace associated to it is maximal [3, 14], namely when the null subspace has the maximum dimension possible. But, what is maximum dimension of a null subspace? When the complex dimension is $n = 2r$, the maximal dimension that a null subspace can be have is r. For example, given the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r, \boldsymbol{e}_{r+1}, \ldots, \boldsymbol{e}_{2r}$, if we define $\boldsymbol{n}_a = \boldsymbol{e}_a + i\,\boldsymbol{e}_{a+r}$, then we have a null subspace of dimension r. In this case we say that it is maximal or maximally null. Therefore, a spinor associated to a null subspace with this dimension is said to be pure. Apart from a multiplicative factor, there is a one-to-one association between $\boldsymbol{\psi}$ and $N_{\boldsymbol{\psi}}$, thus a pure spinor $\boldsymbol{\psi}$ is a representative for $N_{\boldsymbol{\psi}}$ if and only if $\boldsymbol{V}\boldsymbol{\psi} = 0$ for all $\boldsymbol{V} \in N_{\boldsymbol{\psi}}$.

We should recall that a vector $\boldsymbol{V}$ transforms under rotation as $\boldsymbol{R}\boldsymbol{V}\boldsymbol{R}^{-1}$ while the spinor $\boldsymbol{\psi}$ transform as $\boldsymbol{R}\boldsymbol{\psi}$. So, let $\boldsymbol{\psi}$ be a pure spinor and $\boldsymbol{V}$ a null vector that belongs to $N_{\boldsymbol{\psi}}$, then $\boldsymbol{V}\boldsymbol{\psi} = 0$. If $\boldsymbol{V}' = \boldsymbol{R}\boldsymbol{V}\boldsymbol{R}^{-1}$ and $\boldsymbol{\psi}' = \boldsymbol{R}\boldsymbol{\psi}$. It follows that

$$\begin{aligned} \boldsymbol{V}'\boldsymbol{\psi}' &= \boldsymbol{R}\boldsymbol{V}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{\psi} = \boldsymbol{R}\boldsymbol{V}\boldsymbol{\psi} \\ \Longrightarrow \boldsymbol{V}'\boldsymbol{\psi}' &= 0\,. \end{aligned} \tag{2.118}$$

i.e., we can transform any null subspace $N_{\boldsymbol{\psi}}$ into any another null subspace $N'_{\boldsymbol{\psi}}$ by a rotation or reflection [3]. It is worth stressing that the sum of two pure spinors is not a pure spinor in general, since the purity condition is quadratic on the spinor [11, 1]. In practice, given a spinor $\boldsymbol{\psi}$ when is it pure? In order to answer this question, let us workout an example in $\mathbb{R}^3$. Let $S = \{\boldsymbol{\psi} \in \mathcal{C}\ell(\mathbb{R}^3)\,|\,\boldsymbol{\psi} =$

$\psi_1\boldsymbol{\xi}_1 + \psi_2\boldsymbol{\xi}_2 \quad \forall \quad \psi_1, \psi_2 \in \mathbb{C}\}$ be the minimal left ideal of the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^3)$ where $\boldsymbol{\xi}_1 = (1+\hat{\boldsymbol{e}}_3)/2, \boldsymbol{\xi}_2 = \boldsymbol{e}_1\boldsymbol{\xi}_1$ and let $\boldsymbol{V} = V_1\hat{\boldsymbol{e}}_1 + V_2\hat{\boldsymbol{e}}_2 + V_3\hat{\boldsymbol{e}}_3$ be a null vector, *i.e.*, $< \boldsymbol{V}, \boldsymbol{V} >= V_1^2 + V_2^2 + V_3^2 = 0$. Here, the pseudoscalar $\boldsymbol{I} = \hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3$ commutes with all elements and squares equal to -1 and therefore be viewed as the unit imaginary i. Indeed $\boldsymbol{I}\boldsymbol{\xi}_1 = i\boldsymbol{\xi}_1$. Acting $\boldsymbol{V}$ on $\boldsymbol{\psi}$

$$\begin{aligned} \boldsymbol{V}\boldsymbol{\psi} &= (V_1\hat{\boldsymbol{e}}_1 + V_2\hat{\boldsymbol{e}}_2 + V_3\hat{\boldsymbol{e}}_3)(\psi_1\boldsymbol{\xi}_1 + \psi_2\boldsymbol{\xi}_2) \\ &= (V_1\psi_2 + V_3\psi_1 - iV_2\psi_2)\boldsymbol{\xi}_1 + (V_1\psi_1 - V_3\psi_2 + iV_2\psi_1)\boldsymbol{\xi}_2 = 0 \end{aligned} \tag{2.119}$$

we obtain the following system to be solved

$$\begin{cases} V_1\psi_2 + V_3\psi_1 - iV_2\psi_2 &= 0 \\ V_1\psi_1 - V_3\psi_2 + iV_2\psi_1 &= 0 \\ V_1^2 + V_2^2 + V_3^2 &= 0 \end{cases}$$

whose solution is

$$\begin{aligned} V_1 &= {\psi_1}^2 - {\psi_2}^2 \\ V_2 &= i({\psi_1}^2 + {\psi_2}^2) \\ V_3 &= -2\psi_1\psi_2\,. \end{aligned} \tag{2.120}$$

Therefore, if the components of the spinor $\boldsymbol{\psi}$ is characterized by a set of quadratic relations, then the spinor $\boldsymbol{\psi}$ is said to be pure. It turns out that pure spinors are necessarily semi-spinors. Let us prove this. Since the dimension of $\mathcal{V}_{\mathbb{C}}$ is $2r$ we can choose the first r vectors to be the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r\}$ such that $< \boldsymbol{e}_a, \boldsymbol{e}_b >= 0$. Thus, this set forms a basis for a maximal null subspace. To complete this basis we choose the other r as being the set of the vectors $\{\boldsymbol{e}^{r+1}, \boldsymbol{e}^{r+2}, \ldots, \boldsymbol{e}^{2r}\}$ such that $< \boldsymbol{e}^a, \boldsymbol{e}^b >= 0$. Theses two sets form a basis for the whole vector space $\mathcal{V}_{\mathbb{C}}$ and its elements are such that $< \boldsymbol{e}_a, \boldsymbol{e}^b >= \frac{1}{2}\delta_a^b$. The pseudoscalar of $\mathcal{C}\ell_{\mathbb{C}}(\mathcal{V})$ has the following proportionality

$$\begin{aligned} \boldsymbol{I} &\propto (\boldsymbol{e}_1 \wedge \boldsymbol{e}^1)(\boldsymbol{e}_2 \wedge \boldsymbol{e}^2)\ldots(\boldsymbol{e}_r \wedge \boldsymbol{e}^r) \\ &\propto [\boldsymbol{e}_1, \boldsymbol{e}^1][\boldsymbol{e}_2, \boldsymbol{e}^2]\ldots[\boldsymbol{e}_r, \boldsymbol{e}^r]\,, \end{aligned} \tag{2.121}$$

where $[\boldsymbol{e}_a, \boldsymbol{e}^b] = \boldsymbol{e}_a\boldsymbol{e}^b - \boldsymbol{e}^b\boldsymbol{e}_a$ denote the Clifford commutators. Since the vectors $\boldsymbol{e}_a$ span N_ψ, by definition that $\boldsymbol{e}_a\boldsymbol{\psi} = 0$. Hence $\boldsymbol{e}^b\boldsymbol{e}_a\boldsymbol{\psi} = 0$. Thus we only need to know the action of $\boldsymbol{e}_a\boldsymbol{e}^b$ on $\boldsymbol{\psi}$, that is,

$$(\boldsymbol{e}_a\boldsymbol{e}^b)\boldsymbol{\psi} = (\boldsymbol{e}_a\boldsymbol{e}^b + \boldsymbol{e}^b\boldsymbol{e}_a)\boldsymbol{\psi} = 2 < \boldsymbol{e}_a, \boldsymbol{e}^b > \boldsymbol{\psi} = \delta_a^b\boldsymbol{\psi}\,. \tag{2.122}$$

Thus, the action of commutator is

$$[\boldsymbol{e}_a, \boldsymbol{e}^b]\boldsymbol{\psi} = (\boldsymbol{e}_a\boldsymbol{e}^b - \boldsymbol{e}^b\boldsymbol{e}_a)\boldsymbol{\psi} = \delta_a^b\boldsymbol{\psi}\,. \tag{2.123}$$

Therefore, using (2.121) and (2.123) is trivial show that $\boldsymbol{I}\boldsymbol{\psi} \propto \boldsymbol{\psi}$. It follows that every pure spinor $\boldsymbol{\psi}$ must be a semi-spinor, or Weyl spinor. It turns out that in two, four and six dimensions all semi-spinors are pure, while in higher dimensions not all are semi-spinors.

2.4.3 SPin-Invariant Inner Products

We have referred to a minimal left ideal as the spinor space and its elements as spinors, we now examine some operations and properties that this space possess with respect to products of two of its elements. We shall construct an invariant product under the action of the spin group. For example, the space of spinors $S = \mathcal{C}\ell(\mathbb{R}^3)\boldsymbol{\xi}_1$ given by

$$S \;=\; \left\{\boldsymbol{\psi} \in \mathcal{C}\ell(\mathbb{R}^3) \,|\, \boldsymbol{\psi} \;=\; \psi_1\boldsymbol{\xi}_1 \,+\, \psi_2\boldsymbol{\xi}_2 \;\forall\; \psi_1, \psi_2 \in \mathbb{C}\right\}, \tag{2.124}$$

obtained in the Example 3 can be represented as follows:

$$S \simeq \left\{[\boldsymbol{\psi}] \in \mathcal{M}(2,\mathbb{C})\boldsymbol{\xi}_1 \,|\, [\boldsymbol{\psi}] \;=\; \begin{bmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{bmatrix} \quad \forall \quad \psi_1, \psi_2 \in \mathbb{C}\right\}. \tag{2.125}$$

But since the product $\boldsymbol{\xi}_1\boldsymbol{\xi}_2 \;=\; 0$, the set

$$\mathbb{D} \;=\; \boldsymbol{\xi}_1\,\mathcal{C}\ell(\mathbb{R}^3)\,\boldsymbol{\xi}_1 \;=\; \{\psi_1\boldsymbol{\xi}_1\,, \psi_1 \in \mathbb{C}\} \simeq \left\{\begin{bmatrix} \psi_1 & 0 \\ 0 & 0 \end{bmatrix} \,|\, \psi_1 \in \mathbb{C}\right\} \tag{2.126}$$

is a subalgebra of $\mathcal{C}\ell(\mathbb{R}^3)$ with unity $\boldsymbol{\xi}_1$ such that $\boldsymbol{\lambda}\boldsymbol{\xi}_1 \;=\; \boldsymbol{\xi}_1\boldsymbol{\lambda}$, for all $\boldsymbol{\lambda} \in \mathbb{D}$. According to equation (2.40), none of its elements is invertible as an element of $\mathcal{C}\ell(\mathbb{R}^3)$, however there is a unique $\boldsymbol{\lambda}_2 \in \mathbb{D}$ with $\boldsymbol{\lambda}_2\boldsymbol{\lambda}_1 \;=\; \boldsymbol{\xi}_1$ if $\boldsymbol{\lambda}_1$ is a non-zero element of $\mathbb{D}$. Thus, $\mathbb{D}$ is a division algebra and hence the idempotent $\boldsymbol{\xi}_1$ is primitive in $\mathcal{C}\ell(\mathbb{R}^3)$ as expected. The space S has a natural right $\mathbb{D}$-linear structure defined by

$$\begin{array}{rcl} S \times \mathbb{D} & \rightarrow & S \\ \boldsymbol{\psi}, \boldsymbol{\lambda} & \mapsto & \boldsymbol{\psi}\boldsymbol{\lambda} \end{array} \quad . \tag{2.127}$$

The space S endowed with this right $\mathbb{D}$-linear structure is called, more formally, the spinor space. We must note that on spinorial representation an arbitrary multivector $\mathcal{A} \in \mathcal{C}\ell(\mathbb{R}^3)$ and its reverse are given by (see the Example 1):

$$[\mathcal{A}] \;=\; \begin{bmatrix} z_1 & z_3 \\ z_2 & z_4 \end{bmatrix} \quad ; \quad [\widetilde{\mathcal{A}}] \;=\; \begin{bmatrix} z_1^* & z_2^* \\ z_3^* & z_4^* \end{bmatrix} \quad , \; z_i \in \mathbb{C}\,. \tag{2.128}$$

Then, given two spinors $\boldsymbol{\psi}, \boldsymbol{\phi} \in S$ the product

$$[\widetilde{\boldsymbol{\psi}}][\boldsymbol{\phi}] \;=\; \begin{bmatrix} \psi_1^* & \psi_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ \phi_2 & 0 \end{bmatrix} \;=\; \begin{bmatrix} \psi_1^*\phi_1 \,+\, \psi_2^*\phi_2 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{D} \tag{2.129}$$

falls in the division algebra $\mathbb{D}$. Now, the inner product can be naturally introduced on the spinor space as

$$\begin{array}{rcl} S \times S & \mapsto & \mathbb{D} \\ \boldsymbol{\psi}, \boldsymbol{\phi} & \mapsto & (\boldsymbol{\psi}, \boldsymbol{\phi}) \;=\; \widetilde{\boldsymbol{\psi}}\boldsymbol{\phi}\,, \end{array} \tag{2.130}$$

if for $\boldsymbol{\psi} \in S$ and $\boldsymbol{\lambda} \in \mathbb{D}$ the following relation holds:

$$\widetilde{(\boldsymbol{\psi}\boldsymbol{\lambda})} = \widetilde{\boldsymbol{\lambda}}\widetilde{\boldsymbol{\psi}}\,, \tag{2.131}$$

where $\boldsymbol{\lambda} \to \widetilde{\boldsymbol{\lambda}}$ is the complex conjugation of the division algebra $\mathbb{D}$ and the reversion $\boldsymbol{\psi} \mapsto \widetilde{\boldsymbol{\psi}}$ is a semi-linear map. Now, let $\mathcal{A}$ be an arbitrary element of $\mathcal{C}\ell(\mathbb{R}^3)$. Then,

$$(\mathcal{A}\boldsymbol{\psi}, \mathcal{A}\boldsymbol{\phi}) = \widetilde{\mathcal{A}\boldsymbol{\psi}}\mathcal{A}\boldsymbol{\phi} = \widetilde{\boldsymbol{\psi}}\widetilde{\mathcal{A}}\mathcal{A}\boldsymbol{\phi}$$

This implies that the elements $\mathcal{A} \in \mathcal{C}\ell(\mathbb{R}^3)$ such that $\widetilde{\mathcal{A}}\mathcal{A} = 1$ preserve the inner product, *i.e.*, this inner product between spinors is invariant under the action of the group $Spin_+(\mathbb{R}^3)$. In the same way, we can use another involution to define a different inner product. Indeed, the spinorial representation of $\mathcal{A} \in \mathcal{C}\ell(\mathbb{R}^3)$ furnish that its Clifford conjugate is:

$$[\overline{\mathcal{A}}] = \begin{bmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{bmatrix} \quad, z_i \in \mathbb{C}\,. \tag{2.132}$$

We can use this Clifford conjugation to introduce an inner product for spinors. However, the product

$$[\overline{\boldsymbol{\psi}}][\boldsymbol{\phi}] = \begin{bmatrix} 0 & 0 \\ -\psi_2 & \psi_1 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ \phi_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \psi_1\phi_2 - \psi_2\phi_1 & 0 \end{bmatrix} \notin \mathbb{D} \tag{2.133}$$

does not fall in the division algebra $\mathbb{D}$. But since $\overline{\boldsymbol{\psi}}\boldsymbol{\phi} = (\psi_1\phi_2 - \psi_2\phi_1)\boldsymbol{\xi}_2$ its always possible to find an invertible element $\boldsymbol{V} = \hat{\boldsymbol{e}}_1 \in \mathcal{C}\ell(\mathbb{R}^3)$, in this case, such that the product $\hat{\boldsymbol{e}}_1\overline{\boldsymbol{\psi}}\boldsymbol{\phi} = (\psi_1\phi_2 - \psi_2\phi_1)\hat{\boldsymbol{e}}_1\boldsymbol{\xi}_2 = (\psi_1\phi_2 - \psi_2\phi_1)\boldsymbol{\xi}_1$. Thus,

$$[\boldsymbol{V}][\overline{\boldsymbol{\psi}}][\boldsymbol{\phi}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \psi_1\phi_2 - \psi_2\phi_1 & 0 \end{bmatrix} = \begin{bmatrix} \psi_1\phi_2 - \psi_2\phi_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{D}\,. \tag{2.134}$$

This leads us to the conclusion that,

$$\begin{array}{rcll} S \times S & \mapsto & \mathbb{D} & \\ \boldsymbol{\psi}, \boldsymbol{\phi} & \mapsto & (\boldsymbol{\psi}, \boldsymbol{\phi}) & = \boldsymbol{V}\overline{\boldsymbol{\psi}}\boldsymbol{\phi} \end{array} \tag{2.135}$$

also defines an inner product. One can easily prove that this latter inner product between spinors is invariant under the action of the group $Spin(\mathbb{R}^3)$.

In what follows, we will deal with a general Clifford algebra on which every spinor space admits a real, complex or quaternion division algebra. Let η be some involution. If $\boldsymbol{\xi}$ is any primitive idempotent then $\boldsymbol{\xi}^\eta = \boldsymbol{J}\boldsymbol{\xi}\boldsymbol{J}^{-1}$ for some element $\boldsymbol{J}$ with $\boldsymbol{J}^\eta = \varepsilon\boldsymbol{J}$ and $\varepsilon = \pm 1$ [1]. Then, if $\boldsymbol{\psi}, \boldsymbol{\phi} \in S$, we define:

$$\begin{array}{rcll} S \times S & \to & \mathbb{D} & \\ \boldsymbol{\psi}, \boldsymbol{\phi} & \mapsto & (\boldsymbol{\psi}, \boldsymbol{\phi}) & = \boldsymbol{J}^{-1}\boldsymbol{\psi}^\eta\boldsymbol{\phi} \end{array}\,. \tag{2.136}$$

If $\mathcal{A} \in \mathcal{C}\ell_{\mathbb{C}}(\mathcal{V})$ is arbitrary element we find that:

$$(\boldsymbol{\psi}, \mathcal{A}\boldsymbol{\phi}) = (\mathcal{A}^{\eta}\boldsymbol{\psi}, \boldsymbol{\phi})\,. \tag{2.137}$$

Using the equation (2.127) we find that if $\boldsymbol{\lambda} \in \mathbb{D}$,

$$(\boldsymbol{\psi}, \boldsymbol{\phi}\boldsymbol{\lambda}) \;=\; (\boldsymbol{\psi}, \boldsymbol{\phi})\boldsymbol{\lambda}\,, \tag{2.138}$$

the product is $\mathbb{D}$-linear in the second entry. However if we denote $\boldsymbol{\lambda}^{j} \;=\; \boldsymbol{J}^{-1}\boldsymbol{\lambda}^{\eta}\boldsymbol{J}$ for $\boldsymbol{\lambda} \in \mathbb{D}$ it is straightforward to prove that

$$(\boldsymbol{\psi}\boldsymbol{\lambda}, \boldsymbol{\phi}) \;=\; \boldsymbol{\lambda}^{j}(\boldsymbol{\psi}, \boldsymbol{\phi}) \tag{2.139}$$

j is an involution of $\mathbb{D}$ and it will reverse the order of terms in the inner product, that is,

$$(\boldsymbol{\psi}, \boldsymbol{\phi}) \;=\; \varepsilon(\boldsymbol{\phi}, \boldsymbol{\psi})^{j}\,. \tag{2.140}$$

The inner product $(\ ,\ )$ can be $\mathbb{D}^{j}$-symmetric or $\mathbb{D}^{j}$-skew-symmetric depending on the dimension of the space, *i.e.*, when ε is plus or minus, respectively. Let us define the $\mathbb{D}$-linear map f from S to $\mathbb{D}$ by the following relation

$$f(\boldsymbol{\psi})\boldsymbol{\phi} \;=\; (\boldsymbol{\psi}, \boldsymbol{\phi})\,. \tag{2.141}$$

It follows that $f(\boldsymbol{\psi})$ can be naturally identified with $\boldsymbol{J}^{-1}\boldsymbol{\psi}^{\eta}$ by

$$f(\boldsymbol{\psi}) \;=\; \boldsymbol{J}^{-1}\boldsymbol{\psi}^{\eta}\,. \tag{2.142}$$

Note that the function $f(\boldsymbol{\psi})$ depends on the choice of the invertible element $\boldsymbol{J}$.

There are some elements that play an important role when we are dealing with spinors. Let us denote by $\dagger$ the involution[6] of Hermitian conjugation, $*$ the involution of complex conjugate and the composition $\dagger * \equiv t$ stands transpose. So, the invertible matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are related with these involutions by:

$$\boldsymbol{A}\,\boldsymbol{V}\,\boldsymbol{A}^{-1} \;=\; -\boldsymbol{V}^{\dagger} \quad , \quad \boldsymbol{B}\,\boldsymbol{V}\,\boldsymbol{B}^{-1} \;=\; \boldsymbol{V}^{*} \quad , \quad \boldsymbol{C}\,\boldsymbol{V}\,\boldsymbol{C}^{-1} \;=\; -\boldsymbol{V}^{t}\ ,\ \boldsymbol{V} \in \mathcal{C}\ell_{\mathbb{C}}(\mathcal{V}). \tag{2.143}$$

In particular the matrix $\boldsymbol{B}$ enables us define the notion of complex conjugation of a spinor. This is an operation that the spinor space S admits, the so-called **charge conjugation**. This is an anti-linear operation given by

$$\begin{array}{rccll} c: & S & \mapsto & S & \\ & \boldsymbol{\psi} & \mapsto & \boldsymbol{\psi}^{c} & = \boldsymbol{B}^{-1}\boldsymbol{\psi}^{*}\,, \end{array} \tag{2.144}$$

[6] Note that the dagger of the representation of an arbitrary multivetor is the reversion itself.

with the property that

$$(\mathcal{A}\boldsymbol{\psi})^c = \mathcal{A}^*\boldsymbol{\psi}^c \quad \forall\, \mathcal{A} \in \mathcal{C}\ell_{\mathbb{C}}(\mathcal{V}) \text{ and } \boldsymbol{\psi} \in S\,. \tag{2.145}$$

The charge conjugation has different froms depending on the signature and on the dimension [12, 11]. In particular, the conjugate of the conjugate can be equal to identity or equal to minus the identity depending on the signature and on the dimension. The spinors which satisfy

$$\boldsymbol{\psi}^c = \pm\boldsymbol{\psi}\,, \tag{2.146}$$

are called Majorana spinors and the spinors satisfying the above equation and $\boldsymbol{I}\boldsymbol{\psi} = \pm\varepsilon\boldsymbol{\psi}$ simultaneously are called Majorana-Weyl spinors [11, 1]. We can use the matrix C to rewrite an inner product between spinors. If $\boldsymbol{\psi}, \boldsymbol{\phi}$ are two spinors, then

$$(\boldsymbol{\psi}, \boldsymbol{\phi}) = \boldsymbol{\psi}^t \boldsymbol{C}\boldsymbol{\phi} \tag{2.147}$$

is also invariant under, for example, rotation. We can also define other products that are invariant under the action of the connected part to identity of the orthogonal group, for example, rotation and reflection. These are,

$$(\boldsymbol{\psi}, \boldsymbol{I}\boldsymbol{\phi}) \quad ; \quad (\boldsymbol{\psi}^c, \boldsymbol{\phi}) \quad ; \quad (\boldsymbol{\psi}^c, \boldsymbol{I}\boldsymbol{\phi})\,. \tag{2.148}$$

In general, no further inner products can be generated in a trivial way and all these products are diferent and independent [9]. In physics, to make manipulations explicit and less abstract of the Clifford algebra and of the spinorial formalism we use the so-called Dirac matrices. In next section we will explain this point.

2.5 Spinors, Gamma Matrices and their Symmetries

In this section we shall introduce a matrix realization for the elements of the Clifford algebra through the introduction of the Dirac matrices. We will only consider an even dimensional vector space $\mathcal{V}$ of dimension $n = 2r$. These matrices are the representation in dimension n of the Clifford algebra generated by n matrices Γ_a which satisfy the anti-commutation relations

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\delta_{ab}\mathbb{I}\,, \quad a, b = 1, 2, \ldots, n\,, \tag{2.149}$$

where $\mathbb{I}$ is a $2^r \times 2^r$ unit matrix and δ_{ab} is the Kronecker symbol, since repeated multiplication of the Dirac matrices leads to a set of 2^n matrices Γ_A:

$$\Gamma_A : \mathbb{I}, \Gamma_a, \Gamma_{ab}, \Gamma_{abc}, \ldots, \tag{2.150}$$

where,

$$\Gamma_{ab} = \Gamma_{[a}\Gamma_{b]} \quad (a < b), \quad \Gamma_{abc} = \Gamma_{[a}\Gamma_b\Gamma_{c]} \quad (a < b < c), \ldots . \tag{2.151}$$

We can define a set of r raising and lowering operators as:

$$\Gamma_{a\pm} = \frac{1}{2}(\Gamma_a \pm i\,\Gamma_{2a}), \quad a = 1, 2, \ldots, r, \tag{2.152}$$

which satisfy the following anticommutation relations

$$\{\Gamma_{a+}, \Gamma_{a-}\} = \delta_{ab} \tag{2.153}$$
$$\{\Gamma_{a+}, \Gamma_{a+}\} = \{\Gamma_{a-}, \Gamma_{a-}\} = 0 . \tag{2.154}$$

Thus, in particular, note that $(\Gamma_{a\pm})^2 = 0$. Let us first give the usual representation of the Clifford algebra in terms of the Pauli matrices. In two dimensions, $n = 2$, the Pauli matrices $\sigma_a\,(a = 1, 2, 3)$ represent faithfully the Clifford algebra by 2×2 matrices given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} ; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.155}$$

where $\sigma_1\sigma_2 = i\sigma_3$, they square are equal to $\mathbb{I}$ and are hermitian. In this case, we find that

$$\Gamma_{1\pm} = \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\,\sigma_2) . \tag{2.156}$$

If we act Γ_{a-} repeatedly times we eventually arrive in a spinor $\boldsymbol{\xi}^0$ such that

$$\Gamma_{a-}\boldsymbol{\xi}^0 = 0, \quad \forall\, a, \tag{2.157}$$

it is annihilated by all the Γ_{a-}. Since $(\Gamma_{a+})^2 = 0$, we can act Γ_{a+} on the spinor $\boldsymbol{\xi}^0$ over all possible ways at most once each. In order to perform this, let us label the spinor with the indices $s_1, s_2, \ldots, s_r$. If we assume that every index s_a can take the values $1/2$ and $-1/2$, then, starting from $\boldsymbol{\xi}^0$ one obtains a representation of dimension 2^r by action with the elements of the form

$$(\Gamma_{a+})^{s_r+1/2} \ldots (\Gamma_{1+})^{s_1+1/2} . \tag{2.158}$$

This action we denote by

$$\boldsymbol{\xi}^{s_1,s_2,\ldots,s_r} = (\Gamma_{a+})^{s_r+1/2} \ldots (\Gamma_{1+})^{s_1+1/2}\boldsymbol{\xi}^0 . \tag{2.159}$$

Note that $\boldsymbol{\xi}^{-1/2,-1/2,\ldots,-1/2} = \boldsymbol{\xi}^0$ is the our initial spinor. Regarding the spinors, it is useful to define the following column vectors:

$$\xi^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad \xi^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{2.160}$$

In terms these column vectors, the spinor $\boldsymbol{\xi}^0$ is written as

$$\boldsymbol{\xi}^0 = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{r \text{ times}},$$

and therefore the spinor $\boldsymbol{\xi}^{s_1,s_2,\ldots,s_r}$ can be written as

$$\boldsymbol{\xi}^{s_1,s_2,\ldots,s_r} \equiv \boldsymbol{\xi}^{\{s\}} = \boldsymbol{\xi}^{s_1} \otimes \boldsymbol{\xi}^{s_2} \otimes \ldots \otimes \boldsymbol{\xi}^{s_r} . \tag{2.161}$$

Defined this, taking the equation (2.161) as a basis, the matrices Γ_a can be derived from the definitions and the anticommutation relations. For example, when $n = 2$ we find that

$$\sigma_- \boldsymbol{\xi}^{-\frac{1}{2}} = 0 \quad , \quad \sigma_- \boldsymbol{\xi}^{\frac{1}{2}} = \boldsymbol{\xi}^{-\frac{1}{2}} \quad , \quad \sigma_+ \boldsymbol{\xi}^{-\frac{1}{2}} = \boldsymbol{\xi}^{\frac{1}{2}} \quad , \quad \sigma_+ \boldsymbol{\xi}^{\frac{1}{2}} = \boldsymbol{\xi}^{\frac{3}{2}}$$
$$\Longrightarrow \quad \sigma_\pm \boldsymbol{\xi}^s = \boldsymbol{\xi}^{s\pm 1} . \tag{2.162}$$

For more clearness on notation we shall explicit the dimensionality of the Dirac matrices, which can be obtained recursively by the relations

$$\Gamma_1^{(2r)} = \sigma_1 \otimes \mathbb{I}_{2^{(r-1)}} \quad , \quad \Gamma_2^{(2r)} = \sigma_2 \otimes \mathbb{I}_{2^{(r-1)}} \quad ,$$
$$\Gamma_a^{(2r)} = \sigma_1 \otimes \Gamma_{a-2}^{(r-1)} \, , \; a = 3, \ldots, n \, , \tag{2.163}$$

where $\mathbb{I}_N$ is a $N \times N$ unit matrix and $\Gamma_a^{(2r)}$ are $2^r \times 2^r$ matrices. We can split it into those that are even or odd. Explicitly they are

$$\Gamma_{2a-1}^{(2r)} = \underbrace{\sigma_3 \otimes \ldots \otimes}_{(a-1) \text{ times}} \sigma_1 \underbrace{\otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{(r-a) \text{ times}}$$
$$\Gamma_{2a}^{(2r)} = \underbrace{\sigma_3 \otimes \ldots \otimes}_{(a-1) \text{ times}} \sigma_2 \underbrace{\otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{(r-a) \text{ times}} . \tag{2.164}$$

An important fact is that in even dimensions a complete set of $2^r \times 2^r$ matrices is provided by $\Gamma_A = \mathbb{I}, \Gamma_a, \Gamma_{ab}, \ldots$. The $\frac{n(n-1)}{2}$ matrices

$$\Sigma_{ab} = \frac{1}{4} \left[\Gamma_a, \Gamma_b \right] , \quad a = 1, 2, \ldots, n \, , \tag{2.165}$$

satisfy the $SO(n)$ algebra

$$[\Sigma_{ab}, \Sigma_{cd}] = \delta_{bc}\Sigma_{ad} - \delta_{ac}\Sigma_{bd} - \delta_{bd}\Sigma_{ac} + \delta_{ad}\Sigma_{bc} \, , \tag{2.166}$$

and generate a representation of dimension 2^r of $Spin(n)$ the double cover of $SO(n)$. Note that the operator $\Sigma^{a,2a}$ is hermitian and can be therefore simultaneously diagonalized. It simple a manner to prove from definition below

$$S_a \equiv \Sigma_{a,2a} = \Gamma_{a+}\Gamma_{a-} - \frac{1}{2} \, , \tag{2.167}$$

that if the spinor $\boldsymbol{\xi}^{\{s\}}$ is an eigenspinor of the operator S_a its eigenvalues are $s_a = \pm 1/2$. Indeed, if we denote $M_a = \Gamma_{a+}\Gamma_{a-}$, using the anticommutation relations of the raising and lowering operators we find that $(M_a)^2 = \Gamma_{a+}\Gamma_{a-} = M_a$, then $M_a(M_a - 1) = 0$. If $\boldsymbol{\xi}^{\{s\}}$ is also an eigenspinor of the operator M_a with eigenvalue m_a for all a, $M_a(M_a-1)\boldsymbol{\xi}^{\{s\}} = m_a(m_a-1)\boldsymbol{\xi}^{\{s\}}$, it follows that m_a can take just the value, $m_a = 0, 1$. Thus,

$$S_a\boldsymbol{\xi}^{\{s\}} = (M_a - \frac{1}{2})\boldsymbol{\xi}^{\{s\}} = (m_a - \frac{1}{2})\boldsymbol{\xi}^{\{s\}} = s_a\boldsymbol{\xi}^{\{s\}}\,, \tag{2.168}$$

where $s_a = m_a - \frac{1}{2}$ are the eigenvalues of S_a with $\boldsymbol{\xi}^{\{s\}}$ as eigenspinor. Note that,

$$s_a = \begin{cases} \;\;\; 1/2 & , \; m_a = 1 \\ - \; 1/2 & , \; m_a = 0 \end{cases}, \tag{2.169}$$

and the notation $\{s\}$ introduced previously reflects the $Spin(n)$ properties of the spinors [33]. The half-integer values of s_a show that this is a spinor representation. Then, the spinors $\boldsymbol{\xi}^{\{s\}}$ form the Dirac representation of the $Spin(n)$ algebra of dimension 2^r.

The chiral matrix, denoted by Υ, is given by

$$\Upsilon = (-i)^r\Gamma_1\Gamma_2\ldots\Gamma_n = \underbrace{\sigma_3\otimes\ldots\otimes\sigma_3}_{r\ \text{times}}\,, \tag{2.170}$$

and since Υ is not proportional to identity and commute with all Σ_{ab}, it follows that the representation $\boldsymbol{\xi}^{\{s\}}$ of $Spin(n)$ with generators Σ^{ab} must be reducible. Indeed, using (2.163) we find

$$\Sigma_{ab} = \begin{bmatrix} \Sigma^{+}_{ab} & 0 \\ 0 & \Sigma^{-}_{ab} \end{bmatrix}\,, \tag{2.171}$$

where both $\Sigma^{\pm}_{ab}$ satisfy the commutation rules (2.166). Using that $\Gamma_{a+}\Gamma_{a-} - \frac{1}{2} = \Gamma_a\Gamma_{2a}/2$ is straightforward to find

$$\Upsilon = 2^r S_1 S_2 \ldots S_r\,, \tag{2.172}$$

and we see that Υ is diagonal in the basis $\boldsymbol{\xi}^{\{s\}}$. Since $\Upsilon^2 = 1$, using (2.172) and (2.168) we see that the action of Υ on $\boldsymbol{\xi}^{\{s\}}$ is

$$\Upsilon\boldsymbol{\xi}^{\{s\}} = \pm\boldsymbol{\xi}^{\{s\}}\,. \tag{2.173}$$

The 2^{r-1} states with eigenvalue 1 form a Weyl representation of the $Spin(n)$ algebra and the 2^{r-1} states with eigenvalue -1 form a second, inequivalent, Weyl representation. For example, denoting the representation by dimension, for $n = 4$

$$\mathbf{4}_{Dirac} = \mathbf{2} + \mathbf{2}'\,, \tag{2.174}$$

the Dirac representation split into two inequivalent Weyl representation of dimension 2.

Observing that the matrices $\pm\Gamma_{a*}$ satisfy the same Clifford algebra obeyed by Γ_a they must be related to Γ_a by a similarity transformation. In the basis $\boldsymbol{\xi}^{\{s\}}$ the matrices $\Gamma_1, \Gamma_3, \ldots, \Gamma_{n-1}$ are all real while $\Gamma_2, \Gamma_4, \ldots, \Gamma_n$ are all imaginary. This can be viewed directly by means of the equation (2.164). If we define the product of r odd Dirac matrices as the following matrices

$$B_- = \Gamma_1 \Gamma_3 \ldots \Gamma_{n-1} \quad , \quad B_+ = \Upsilon \, B_- \, , \tag{2.175}$$

what is the final result of the operations $B_- \Gamma_a B_-^{-1}$ and $B_+ \Gamma_a B_+^{-1}$? Let us suppose that Γ_a is imaginary, hence its is an even element. So, by anticommutation

$$B_- \Gamma_a B_-^{-1} = (-1)^r \Gamma_a = (-1)^{r-1}(-\Gamma_a) = (-1)^{r-1}\Gamma_a^* \, . \tag{2.176}$$

Now, if Γ_a is real, hence its is an odd element. So,

$$B_- \Gamma_a B_-^{-1} = (-1)^{r-1}\Gamma_a = (-1)^{r-1}\Gamma_a^* \, . \tag{2.177}$$

Thus, for any element Γ_a, be it even or odd, we have

$$B_- \Gamma_a B_-^{-1} = (-1)^{r-1}\Gamma_a^* \, . \tag{2.178}$$

In the same way we can easily prove that

$$B_+ \Gamma_a B_+^{-1} = (-1)^{r}\Gamma_a^* \, . \tag{2.179}$$

Performing similar manipulations we obtain the following results,

$$B_\pm \Sigma_{ab} B_\pm^{-1} = \Sigma_{ab}^* \tag{2.180}$$

This relation implies that the charge conjugation of a spinor $\boldsymbol{\psi}^c = B^{-1}\boldsymbol{\psi}^*$ transforms in the same way as $\boldsymbol{\psi}$. The action of $B^\pm$ on the chirality matrix results

$$B_\pm \Upsilon B_\pm^{-1} = (-1)^{r-1}\Upsilon^* \, , \tag{2.181}$$

from which we can see that the charge conjugation can change the eigenvalue of Υ depending on the parity of $r-1$. For example, when r is odd ($n = 2, 6, 10, 14, 18, \ldots$ or $n = 2 \bmod 4$) each Weyl representation is its own conjugate, while that for r even ($n = 0, 4, 8, 12, 16, \ldots$ or $n = 0 \bmod 4$) each Weyl representation is conjugate to the other. In particular, when $n = 4$

$$\mathbf{4}_{Dirac} = \mathbf{2} + \bar{\mathbf{2}} \, , \tag{2.182}$$

where $\bar{\mathbf{2}}$ is the conjugate of the representation $\mathbf{2}$. A Majorana spinor is an eigenspinor of the charge conjugation operation

$$\boldsymbol{\psi}^c = \pm\boldsymbol{\psi} \, , \tag{2.183}$$

which implies that taking the conjugate $\boldsymbol{\psi}^* = B\boldsymbol{\psi} \therefore (\boldsymbol{\psi}^*)^* = (B\boldsymbol{\psi})^* = B^*\boldsymbol{\psi}^* = B^*B\,\boldsymbol{\psi}$ gives a consistent condition if $B^*B = \mathbb{I}$. But using the reality and anticommutation properties of Γ we find

$$B_-^*B_- = (-1)^{r(r-1)/2}\mathbb{I} \quad , \quad B_+^*B_+ = (-1)^{(r-1)(r-2)/2}\mathbb{I}\,. \tag{2.184}$$

If we use B_- to define the Majorana spinor, then this operation is possible when $r = 1 \pmod 4$ or $r = 4 \pmod 4$ while using B^+ only if $r = 1 \pmod 4$ or $r = 2 \pmod 4$. The such so-called Majorana-Weyl spinors are the spinors that are Majorana and Weyl. In this case, the Majorana condition and Weyl condition only can be imposed if the Weyl representation is conjugate to itself. In particular when r is even it is not possible to impose both the Majorana and Weyl conditions on a spinor. However, when $r = 1 \pmod 4$ a spinor can simultaneously satisfy the Majorana and Weyl conditions.

The Γ-matrices are all hermitian, thus we get the hermiticity property $\Gamma_a^\dagger = \Gamma_a$. So, the operation

$$A\,\Gamma_a A^{-1} = -\Gamma_a^\dagger \tag{2.185}$$

is satisfied if $A\,\Gamma_a = -\Gamma_a A$. The chirality matrix Υ anti-commute with the Dirac matrices. It follows that, in even dimension, we can choose $A = \Upsilon$.

There is always a charge conjugation matrix C , such that

$$C_\pm\,\Gamma_a C_\pm^{-1} = \mp\Gamma_a^t \quad ; \quad C^t = \pm C\,. \tag{2.186}$$

In the representation (2.163) the two possibilities are the following:

$$C_+ = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \dots \tag{2.187}$$

$$C_- = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \dots \tag{2.188}$$

where $C_- \propto C_+\Upsilon$ and $C = C^\dagger = C^{-1}$. By means of the matrix C we define invariant inner products between spinors as previously seen.

3. Curved Spaces and Some Geometrical Aspects of Spinors

The Euclidean vector space, $\mathbb{R}^n$, is an n-dimensional vector space endowed with a positive-definite symmetric metric whose components are $\delta_{\mu\nu}$. But, there are other spaces which we intuitively think of as curved on which we would like to perform similar operations to those at $\mathbb{R}^n$: the sphere S^2, for example. To treat of spaces that may be curved and may have a complicated topology we introduce the notion of manifolds, which are generalizations of our familiar ideas about curves and surfaces to arbitrary dimensional objects. In this chapter, we describe the basic elements of differential geometry which is the ideal tool to deal with manifolds. As we shall see later, Killing vectors, which characterize the symmetries of a manifold, can be used to construct conserved quantities along geodesics that are curves of minimum length connecting two points on a manifold. Moreover, we will find the curvature of a manifold, more precisely of a tangent bundle, is measured by the Riemann tensor which arises from the commutator of the covariant derivatives. The covariant derivative, in formal jargon, is called a connection of the tangent bundle. In particular, we shall uniquely extend it to the spinorial bundle whose sections are the so-called spinor fields.

3.1 Manifolds and Tensors Fields

Manifolds are one of the most fundamental concepts in mathematics and physics. Intuitively, we can think on an n-dimensional manifold M, essentially, as a space that that may be curved, but in local regions looks like Euclidean space, $\mathbb{R}^n$. For example, a curve in three-dimensional Euclidean space is parametrized locally by a single number τ as $(x(\tau), y(\tau), z(\tau))$ and a surface can be parametrized by two numbers u and v as $(x(u,v), y(u,v), z(u,v))$. A curve is a one dimensional manifold since it locally looks like $\mathbb{R}$, while a surface is 2-dimensional manifold since it looks like $\mathbb{R}^2$. In particular, the 2-sphere is a 2-dimensional manifold. The local character enables us to associate to each point on a manifold a set of n numbers called the local coordinate. In general, a manifold may be different from $\mathbb{R}^n$ globally, then it is possible that a single point has two or more coordinates, since generally we have to introduce several local coordinates in order to cover the whole manifold. To develop the usual calculus on a manifold we require that the transition from one coordinate to the other be smooth and the entire manifold be constructed by smoothly sewing together the local regions. More precisely, a manifold of dimension n is a topological set such that the neighborhood of each point can be mapped into a patch of $\mathbb{R}^n$ by a coordinate system in a way that the overlapping neighborhoods are consistently joined [12, 18, 16].

The concept of a vector space is well known to most readers, but when we consider curved geometries the vector space structure is lost. For example, there is no trivial way of add two points on the 2-sphere and end up with a third point on the 2-sphere. Nevertheless, we can recover this structure in the limit of infinitesimal displacements. Imagine curves passing through a point p belonging to the surface of the 2-sphere. The possible directions that these curves can take generate a plane, this is called the tangent space of p. The intuitive notion of a tangent vector at a point $p \in M$ of a n-dimensional manifold is a vector lying in the tangent plane at point p. These vectors generate a vector space of dimension n, denoted by T_pM, called the tangent space of p. Note that, the term vector refer to a vector at a given point p of M. The term vector field, denoted by $\boldsymbol{V}$, refers to a rule for defining a vector at each point of M. The vector fields generate a new $2n$-dimensional manifold, denoted by TM, formed by the union of the tangent spaces of all points of the manifold M,

$$TM \equiv \bigcup_{p\in M} T_pM\,, \tag{3.1}$$

this is called the tangent bundle. Now, consider the curves λ defined in the coordinates x^μ by the equation

$$x^\mu(\tau) \;=\; x^\mu(p) + V^\mu\tau \quad (\mu = 1, 2, \ldots, n)\,, \tag{3.2}$$

for τ in some small interval $-\epsilon < \tau < \epsilon$. Since a curve has a unique parameter, to every curve there is a unique set $\{V^\mu\}$ with

$$V^\mu \;=\; \frac{dx^\mu}{d\tau}\,, \tag{3.3}$$

evaluated at $\tau \;=\; 0$ which are the components of a vector field $\boldsymbol{V}$ tangent to the curve $x^\mu(\tau)$ on a coordinate system $\{x^\mu\}$ in the neighborhood of $p \in M$. The curves $x^\mu(\tau)$ are called the integral curves of the vector field $\boldsymbol{V}$. In this coordinate system, a vector field $\boldsymbol{V}$ has the following abstract notation:

$$\boldsymbol{V} \;=\; V^\mu\,\frac{\partial}{\partial x^\mu} \;\equiv\; V^\mu\,\partial_\mu\,. \tag{3.4}$$

Sometimes, in a particular problem it might be more convenient to use other coordinate system: spherical coordinate, for example. This is simple to implement. For example, under coordinate transformation $x^\mu \mapsto x'^\mu(x^\nu)$ we find that the components of the vector field transform as:

$$\boldsymbol{V} = V^\mu \partial_\mu = V^\mu \left(\frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu\right) \Rightarrow V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu , \tag{3.5}$$

where V'^μ are the components of $\boldsymbol{V}$ on the coordinate system $\{x'^\mu\}$. Note that if f belongs to $\mathfrak{F}(M)$, the space of the functions over the manifold M, then $\boldsymbol{V}(f) = V^\mu \partial_\mu f$ is the derivative of the function f along of the curve whose tangent vector field is $\boldsymbol{V}$ and satisfies the Leibnitz rule when operating on products of functions. Therefore, the vector fields on a manifold define differential operators that act on the space of the functions over the manifold. In addition, given a local coordinate system $\{x^\mu\}$, then the coordinates define the coordinate frame $\{\partial_\mu\}$ for the tangent space at each point. For example, if the $\{x^\mu\} = \{\theta, \phi\}$ is the usual coordinate system on the 2-sphere, then it defines the coordinate frame $\{\partial_\theta, \partial_\phi\}$, with θ being the polar angle and ϕ the azimuthal angle.

Since T_pM is a vector space, there exists a dual vector space to T_pM, denoted by T_p^*M, called the co-tangent space at p, whose elements are the linear functionals from T_pM to $\mathbb{R}$, see the section 2.1. A linear functional is called a co-vector, but in the context of differential forms we always refer to it as 1-form at $p \in M$. The space of the 1-form fields is the space TM^*, defined in analogy to construction of TM, by

$$TM^* \equiv \bigcup_{p\in M} T_p^*M , \tag{3.6}$$

called the co-tangent bundle. The simplest example of a 1-form field is the differential df of a function $f \in \mathfrak{F}(M)$ which, in terms of the coordinate $\{x^\mu\}$, is expressed as $df = (\partial f/\partial x^\mu)dx^\mu$. Note that by the equation (2.3), taking $\boldsymbol{e}^\mu = dx^\mu$ and $\boldsymbol{e}_\mu = \partial_\mu$, it follows that:

$$\boldsymbol{e}^\mu(\boldsymbol{e}_\nu) = dx^\mu(\partial_\nu) = \delta^\mu{}_\nu .$$

Thus, it is fruitful to regard that the differentials $\{dx^\mu\}$ provide a local coordinate frame for the co-tangent space T_p^*M, which is the dual to the local coordinate basis provided by the differential operators $\{\partial_\mu\}$ for the tangent space T_pM. An arbitrary 1-form field $\boldsymbol{\omega}$ is written as:

$$\boldsymbol{\omega} = \omega_\mu \, dx^\mu , \tag{3.7}$$

where the $\omega_\mu = \boldsymbol{\omega}(\partial_\mu)$ are the components of $\boldsymbol{\omega}$. Now, if we pass from the coordinate system $\{x^\mu\}$ to the coordinate system $\{x'^\mu\}$, we easily find that:

$$\boldsymbol{\omega} = \omega_\mu \, dx^\mu = \omega_\mu \left(\frac{\partial x^\mu}{\partial x'^\nu} dx^\nu\right) \Rightarrow \omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu . \tag{3.8}$$

The bases $\{\partial_\mu\}$ and $\{dx^\mu\}$ are sometimes referred to as the canonical bases for the tangent and the cotangent spaces. Notice, in particular, that for every $f \in \mathfrak{F}(M)$ we define the element $df \in T^*M$ by the relation $df(\boldsymbol{V}) = \boldsymbol{V}(f)$. Indeed, using (3.7), it is immediate see that the action of df on $\boldsymbol{V} \in TM$ is given by

$$df(\boldsymbol{V}) = V^\mu \, \frac{\partial f}{\partial x^\mu} = \boldsymbol{V}(f)\,. \tag{3.9}$$

The equations (3.5) and (3.8) express two types of behavior under the arbitrary coordinate transformation $x^\mu \mapsto x'^\mu(x^\nu)$. In general, if $T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}$ are the components of a tensor field $\boldsymbol{T}$ on the coordinate system $\{x^\mu\}$, then we shall represent it on the canonical basis as follows:

$$\boldsymbol{T} \;=\; T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}\, \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_p} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_q}\,. \tag{3.10}$$

Its components on the coordinates $\{x'^\mu\}$ are thus related to its components on the coordinates system $\{x^\mu\}$ by the following tensor transformation law

$$T'^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \;=\; \left(\frac{\partial x'^{\mu_1}}{\partial x^{\sigma_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\sigma_p}}\right)\left(\frac{\partial x^{\rho_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x^{\rho_q}}{\partial x^{\nu_q}}\right) T^{\sigma_1 \ldots \sigma_p}{}_{\rho_1 \ldots \rho_q}\,, \tag{3.11}$$

where $T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \;=\; \boldsymbol{T}(dx^{\mu_1}, \ldots, dx^{\mu_p}, \partial_{\nu_1}, \ldots, \partial_{\nu_q})$. The differential forms are a relevant class of tensors which has all indices down and totally skew-symmetric. For instance, $\omega_{\mu_1 \ldots \mu_p} = \omega_{[\mu_1 \ldots \mu_p]}$ is called a p-form and space of all p-form fields over a manifold M is denoted $\wedge^p M$. Note that we can use several results obtained in the sections 2.1 and 2.2 by letting $\boldsymbol{e}^\mu \;=\; dx^\mu$. In particular, a general p-form field can be written as:

$$\boldsymbol{F} \;=\; \frac{1}{p!}\, F_{\mu_1 \ldots \mu_p}\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}\,,$$

where $\wedge$ is the exterior product. In this language, functions on the manifold are called of 0-forms.

In the section 2.2, we introduced the Clifford algebra as an algebra defined on vector spaces endowed with a symmetric non-degenerate inner product. A metric $\boldsymbol{g}$ on a manifold M is a tensor field that provides an inner product on the tangent space at each point $p \in M$, *i.e.*, it is a map which takes two vector fields and yield a function over the manifold. For example, given two vector fields $\boldsymbol{V}, \boldsymbol{U} \in T_pM$, we have that

$$\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{U}) \;= <\boldsymbol{V}, \boldsymbol{U}> =\; g_{\mu\nu}\, V^\mu\, U^\nu\,, \tag{3.12}$$

where $g_{\mu\nu} \;=\; \boldsymbol{g}(\partial_\mu, \partial_\nu)$ are the components of $\boldsymbol{g}$ in coordinate frame. We can expand the metric $\boldsymbol{g}$ in terms of its components in the coordinate frame as

$$\boldsymbol{g} \;=\; g_{\mu\nu}\, dx^\mu \otimes dx^\nu\,. \tag{3.13}$$

Sometimes, the notation ds^2 is used in the place of $\boldsymbol{g}$ to represent the metric tensor

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu\,. \tag{3.14}$$

In such case it is called a line element. Intuitivelty, the line element ds^2 tells us the infinitesimal distance between two point on manifold. In physics, we almost always assume that the manifold is endowed with a metric, hence the pair $(M, \boldsymbol{g})$ will sometimes be called the manifold. In particular, the model of a spacetime in general relativity is a differential manifold with a metric $\boldsymbol{g}$ on it [18, 16, 19, 20, 22]. A good illustration is the Minkowski manifold, $(\mathbb{R}^4, \boldsymbol{\eta})$, the manifold of special theory of relativity. According to this theory, we live on the vector space $\mathbb{R}^4$ endowed with the metric $ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = dt^2 - dx^2 - dy^2 - dz^2$, where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and $\{x^\mu\} = \{t, x, y, z\}$ are the cartesian coordinates. These coordinates define a coordinate frame $\{\partial_\mu\} = \{\partial_t, \partial_x, \partial_y, \partial_z\}$ in which the components of the metric are ± 1. Actually, it is always possible at some point $p \in M$, by a convenient choice of the coordinates, to put any metric $\boldsymbol{g}$ in a canonical form. In this form the metric components become

$$g_{\mu\nu} = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1, 0, 0, \ldots, 0)\,, \tag{3.15}$$

where the term diag means that $g_{\mu\nu}$ is diagonal matrix with the given elements. If any of the slots are zero, the metric is degenerate, and its inverse will not exist, otherwise it is non-degenerate. In this last case we can define an inverse metric via $g^{\mu\sigma}\, g_{\sigma\nu} = \delta^\mu{}_\nu$ which allows us to raise and lower indices. In its canonical form, we call signature s of the metric the modulus of its trace, $s = |\sum_\mu g_{\mu\mu}|$. Denoting by n the dimension of the manifold then when $s = n$ the metric is said to be Euclidean, for $s = n-2$ the signature is Lorentzian and $s = 0$ the metric is said to have split signature. In what follows, when we refer about a manifold $(M, \boldsymbol{g})$ it will always be assumed that the metric $\boldsymbol{g}$ is non-degenerate.

Since the tensor fields are geometric objects which are well defined independently of the frame chosen, any set of equations built from tensors will keep its physical meaning under arbitrary coordinate transformations. But there is one important object whose transformation under coordinate transformation is not trivial, the derivative of tensors. The partial derivative of a scalar function, $\partial_\mu f$, is a tensor. However, consider the action of the partial derivative ∂_ν in the components of a vector field V^μ on the coordinate system $\{x^\mu\}$, $\partial_\nu V^\mu$. Under a change of coordinates, using the equations (3.5) and (3.8), this becomes

$$\partial'_\nu\, V'^\mu = \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\partial_\sigma V^\rho + \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\rho}\, V^\rho\,, \tag{3.16}$$

obviously, this combination does not transform as a tensor under a arbitrary coordinate transformation, since it is not in accordance with the equation (3.11). This leads to a natural question: how do we form the derivative of a vector V^μ in such a way that the resulting object is a tensor? To overcome this problem, we define a connection called Levi-Civita connection or Christoffel symbol

$$\Gamma^{\mu}{}_{\nu\rho} \equiv \frac{1}{2}\, g^{\mu\sigma} \left(\partial_{\nu} g_{\rho\sigma} + \partial_{\rho} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho}\right) , \tag{3.17}$$

with $g^{\mu\nu}$ being the inverse of $g_{\mu\nu}$. This symbol is used to correct the non-tensorial character of the partial derivative. Indeed, after some algebra, it can be proved that the combination

$$\nabla_{\nu} V^{\mu} \equiv \partial_{\nu}\, V^{\mu} \,+\, \Gamma^{\mu}{}_{\nu\rho}\, V^{\rho} , \tag{3.18}$$

does transform as tensor, *i.e.*,

$$\nabla'_{\nu} V'^{\mu} = \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \nabla_{\sigma} V^{\rho} . \tag{3.19}$$

To ensure that the covariant derivative transforms as a tensor it must transform as

$$\Gamma'^{\mu}{}_{\nu\rho} = \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \frac{\partial x^{\delta}}{\partial x'^{\rho}} \Gamma^{\sigma}{}_{\gamma\delta} - \frac{\partial^2 x'^{\mu}}{\partial x^{\gamma} \partial x^{\delta}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \frac{\partial x^{\delta}}{\partial x'^{\rho}} . \tag{3.20}$$

Note that, this is not, of course, a tensor transformation law. To prove that this occurs, it is enough to use the law of transformation of the metric and the partial derivative. Thus, the Christoffell symbol is not a tensorial object. They are deliberately constructed to be non-tensorial, but in such a way that the combination (3.18) transforms as a tensor. The operator ∇_{ν} in (3.18) is called the covariant derivative. This operator shares many properties with the usual partial derivative, it is linear and obeys the Leibniz rule. Particularly, the latter rule can be used to obtain the expression for covariant derivative of a 1-form $\boldsymbol{\omega}$. In order to perform this, it is enough notice that the action of $\boldsymbol{\omega}$ on $\boldsymbol{V}$ provides a scalar, namely, $\boldsymbol{\omega}(\boldsymbol{V}) = \omega_{\mu} V^{\mu}$. Then, acting the operator ∇_{μ} in this scalar, we are left with the following equation

$$\nabla_{\nu} \omega_{\mu} \equiv \partial_{\nu}\, \omega_{\mu} \,-\, \Gamma^{\rho}{}_{\nu\mu}\, \omega_{\rho} . \tag{3.21}$$

It is worth note that the same Christoffel symbol was used as for the covariant derivative of a vector, but now with a minus sign. In general, for an arbitrary tensor field $\boldsymbol{T}$ whose components on the coordinate system are $T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}$, for each upper indice we introduce a term with a $+\Gamma$, and for each lower indice a term with a $-\Gamma$:

$$\begin{aligned} \nabla_{\sigma} T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} &= \partial_{\sigma} T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \\ &+ \Gamma^{\mu_1}{}_{\sigma\rho}\, T^{\rho\, \mu_2 \ldots i_p}{}_{\nu_1 \ldots \nu_q} + \Gamma^{\mu_2}{}_{\sigma\rho}\, T^{\mu_1 \rho\, \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} + \ldots \\ &- \Gamma^{\rho}{}_{\sigma\nu_1}\, T^{\mu_1 \ldots \mu_p}{}_{\rho\, \nu_2 \ldots \nu_q} - \Gamma^{\rho}{}_{\sigma\nu_2}\, T^{\mu_1 \ldots \mu_p}{}_{\nu_1\, \rho \ldots \nu_q} - \ldots . \end{aligned} \tag{3.22}$$

One might expect that no reference frame is better than another, all of them are equally arbitrary. In particular, if a tensor vanishes in a coordinate system, then it must vanishes identically in any coordinate system. A tensor field $\boldsymbol{T}$ is said to be covariantly constant if its covariant derivative vanishes. In its turn, using the latter formula it is straightforward to prove that $\nabla_\sigma g_{\mu\nu} = 0$. Besides, the concept of moving a tensor along a path, keeping constant all the while, is known as parallel transport. A tensor field $\boldsymbol{T}$ is thus said to be parallel transported along of a curve λ whose tangent vector field is $\boldsymbol{V}$ if the equation

$$V^\sigma \nabla_\sigma T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} = 0\,, \tag{3.23}$$

is satisfied along the curve. The notion of parallel transport is obviously dependent on the connection, and different connections lead to different answers. If the connection is metric-compatible, the metric is always parallel transported with respect to it.

With parallel transport defined, the next logical step is to discuss geodesics. In Euclidean space, the path of shortest distance is the straight line. When we work with curved manilfods, the generalization of the notion of a straight line in Euclidean space is provides by the so-called geodesics. For example, suppose that the components of the tangent vector to curve $x^\mu(\tau)$ on the coordinate system $\{x^\mu\}$ is $T^\mu = dx^\mu/d\tau$ at $\tau = 0$. The condition that it be parallel transported is thus

$$T^\mu \nabla_\mu T^\mu = 0 \quad , \quad T^\mu = dx^\mu/d\tau\,. \tag{3.24}$$

Using (3.18), its straightforward to derive the following differential equation:

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0\,, \tag{3.25}$$

known as the geodesic equation. It is the curve of minimum length connecting two points p_1 and p_2 on a manifold $(M, \boldsymbol{g})$. In other words, it is the curve whose tangent vector T^μ is parallel transported along itself. If, however, we had found a curve on which $T^\mu \nabla_\mu T^\mu$ is proportional to the tangent vector T^μ

$$T^\mu \nabla_\mu T^\nu = f(\tau)\, T^\nu \quad , \quad T^\mu = dx^\mu/d\tau$$

with $f = f(\tau)$ being some function on curve, it is easy to show that we can always reparametrize this curve so that it satisfies the equation (3.24). Thus, without loss of generality we can consider just such a curve. The parameters τ' such that $f = 0$ are said to be an affine parameters. In this case, we would be free to rescale the parameter $\tau \to \tau' = a\tau + b$ with $a \neq 0$ and b being constant.

To define the covariant derivative we need to put an additional structure on our manifold: a connection, which is specified in some coordinate system by a set of n^3 Christoffel symbols. However, there exists a relevant differential operation on the space of the forms which does not depends on such a connection, the so-called exterior differentiation, d. Exterior differentiation is effected by an operator

d applied to forms. This operator maps p-forms into $(p+1)$-forms as:

$$d\boldsymbol{\omega} = \frac{1}{p!}\,\partial_\nu \omega_{\mu_1\ldots\mu_p}\,dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\,. \tag{3.26}$$

Note that the differential df of a function $f \in \mathfrak{F}(M)$ is exactly the exterior derivative of a 0-form. The reason why the exterior derivative is so relevant is that it is a tensor, even in curved manifolds, although we have used the partial derivative. Note also that we could have used the covariant derivative, since Christoffel symbols are symmetric in its indices down $\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}$. The exterior derivative has some important properties. It obeys a modified version of the Leibniz rule when applied to the product of a p-form $\boldsymbol{\omega}$ and a q-form $\boldsymbol{\phi}$:

$$d(\boldsymbol{\omega}\wedge\boldsymbol{\phi}) \,=\, d\boldsymbol{\omega}\wedge\boldsymbol{\phi} \,+\, \widehat{\boldsymbol{\omega}}\wedge d\boldsymbol{\phi}\,, \tag{3.27}$$

where $\widehat{\boldsymbol{\omega}} = (-1)^p\boldsymbol{\omega}$ is the degree involution of the p-form $\boldsymbol{\omega}$. Another remarkable property of the exterior derivative, which is a consequence of the fact that partial derivatives commute, is that, for any form $\boldsymbol{\omega}$,

$$d(d\boldsymbol{\omega}) \,=\, 0\,. \tag{3.28}$$

A p-form are said to be closed if $d\boldsymbol{\omega} = 0$, and exact if $\boldsymbol{\omega} = d\boldsymbol{\phi}$. Clearly, all exact forms are closed, but not all closed forms are exact.

3.2 The Curvature Tensor

Since coordinates are physically meaningless we should always work with tensorial objects, because they have well-defined geometrical meaning. The Christoffel symbol is not a tensor, then it cannot be used as a measure of how much our manifold is curved. For example, in usual vectorial calculus, the laplacian operator ∇^2 in cartesian coordinates looks different than in spherical coordinates. This happens because the laplacian operator is written as $\nabla^2 = \nabla_\mu\nabla^\mu$, it means that we are implicitly using the covariant derivative, but when the cartesian coordinate is employed the Christoffel symbols vanishes and the covariant derivative reduces to partial derivative. Notice, in particular, that the connection defined by the equations (3.17) and (3.22) is named the Levi-Civita connection. The curvature is quantified by the Riemann tensor, which is derived from the connection. Let us investigate the curvature tensor formed from an arbitrary connection, not necessarily Levi-Civita, that is, a connection transforming as (3.20) under a change of coordinates. Though the covariant derivative and the partial derivative has some common properties, there exist a big difference between them. The partial derivatives always commute, while the covariant derivative, in general, do not and non-commutativity of the covariant derivatives implies that the manifold is curved. Then, nothing is fairer than trying to characterize the curvature of a manifold by the commutator of the covariant derivatives. Consider the vector field $\boldsymbol{V}$, after some algebra, one can be proved that:

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \\ &= R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho \,, \end{aligned} \tag{3.29}$$

where

$$R^\rho{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\kappa\mu}\Gamma^\kappa_{\sigma\nu} - \Gamma^\rho_{\kappa\nu}\Gamma^\kappa_{\sigma\mu} \,. \tag{3.30}$$

$$T^\lambda{}_{\mu\nu} \equiv 2\Gamma^\lambda_{[\mu\nu]} \,. \tag{3.31}$$

The objects $T^\lambda{}_{\mu\nu}$ are the components of a tensor called the torsion tensor. Since the left hand side of the equation is a tensor, the object $R^\rho{}_{\sigma\mu\nu}$, although its definition was made in terms of a non-tensorial connection symbol, is also a tensor. This is known as the Riemann tensor. By means of the equation (3.29), we see that the Riemann tensor measures that part of the commutator of covariant derivatives that is proportional to the vector field, while the torsion tensor measures the part that is proportional to the covariant derivative of the vector field. The Riemann tensor is also called the curvature tensor, because it is the measure of the curvature of tangent bundle endowed with a connection. In a curved manifold M, in every point $p \in M$ it is always possible choose a coordinate system on which $g_{\mu\nu} = \delta_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\rho} = 0$ at this point, this is called Riemann normal coordinates. Besides, when the Riemann tensor vanishes we can always construct a coordinate system in which the metric components are constant [19, 20, 23]. In particular, a manifold is said to be flat if, and only if, the Riemann tensor vanishes.

In n dimensions, a general tensor with s indices has n^s independent components. But, if it has symmetries not all these components are independent. For example, it is immediate see that the torsion and Riemann tensors have symmetries, that are $T^\lambda{}_{\mu\nu} = -T^\lambda{}_{\nu\mu}$ and $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$. To find the additional symmetries is useful to examine the Riemann tensor with all low indices. Then, defining $R_{\rho\sigma\mu\nu} = g_{\rho\lambda}R^\lambda{}_{\sigma\nu\mu}$, one can prove that this tensor possesses the following symmetry properties:

$$R_{\rho\sigma\mu\nu} = R_{[\rho\sigma][\mu\nu]} \quad ; \quad R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \,, \tag{3.32}$$

$$R_{\rho[\sigma\mu\nu]} = 0 \quad ; \quad \nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \,. \tag{3.33}$$

These relations are the complete set of symmetries of the Riemann tensor. In particular, the equation (3.33) is known as the Bianchi identities. These symmetries reduce the number of independent components of the Riemann tensor. Indeed, in n dimensions, instead of n^4, it has $\frac{1}{12}n^2(n^2-1)$ independent components. For instance, in four dimensions, the Riemann tensor has 20 independent components. There are other important tensors which are constructed out of the Riemann tensor. Let us decompose this tensor into a trace part and a traceless part. By the antisymmetry properties, the trace of the Riemman tensor over its first two or last indices vaninhes. However, the following trace is non-vanishing

$$R_{\mu\nu} \equiv g^{\rho\sigma} R_{\sigma\mu\rho\nu} = R^{\rho}{}_{\mu\rho\nu} . \tag{3.34}$$

This is called the Ricci tensor. In particular, as a consequence of the symmetries of the Riemann tensor, the Ricci tensor associated with the Levi-Civita connection is automatically symmetric

$$R_{\mu\nu} = R_{\nu\mu} . \tag{3.35}$$

The curvature scalar, R, is defined as the trace of the Ricci tensor:

$$R \equiv g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu} . \tag{3.36}$$

The Ricci tensor and the curvature scalar contain all of the information about traces of the Riemann tensor. The traceles part is called the Weyl tensor. It is given in dimension $n \geq 3$ by

$$C_{\rho\sigma\mu\nu} \equiv R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)} R\, g_{\rho[\mu}g_{\nu]\sigma} . \tag{3.37}$$

This tensor is basically the Riemann tensor with all of its contractions removed and vanishes for a flat manifold. Particularly, in dimensions 3 the Weyl tensor vanishes identically, while in dimensions $n \geq 4$ it is generally non-null. One of the most important properties of the Weyl tensor is that it is invariant under conformal transformations [21]. For instance, if in the neighborhood of a point p of a manifold $(M, \boldsymbol{g})$,

$$g_{\mu\nu} = \Omega^{-2}\delta_{\mu\nu} \quad \Omega \in \mathfrak{F}(M) , \tag{3.38}$$

then $(M, \boldsymbol{g})$ is said to be a conformally flat manifold. Then, by the conformal invariance of the Weyl tensor, it vanishes on $(M, \boldsymbol{g})$. This tensor is one among the most relevant tensors in general relativity. It can be associated to it an algebraic classification and relate such classification with integrability properties of the field equations. For instance, the Petrov classification is an algebraic classification for the Weyl tensor of a 4-dimensional curved manifold which help us on the search of exact solutions for Einstein's equation, the most relevant example being the Kerr metric [23, 12, 25, 27].

3.3 Non-Coordinate Frame and Cartan's Structure Equations

By what was seen previously, the tangent space T_pM and its dual T_p^*M are spanned by the coordinate frames $\{\partial_\mu\}$ and $\{dx^\mu\}$, respectively. However, when the manifold

M is endowed with a metric $\boldsymbol{g}$ it is convenient to use a non-coordinate frame. An element of this frame is a vector field given by the following linear combination,

$$\boldsymbol{e}_a = e_a{}^{\mu}\,\partial_\mu\,, \tag{3.39}$$

where the index $a = 1, 2, \ldots, n$ is only a label for the n vector fields, it is not a tensorial index. Moreover, the coefficients of the linear combination are such that $\det e_a{}^{\mu} \neq 0$. Under the metric $\boldsymbol{g}$, we desire that $\{\boldsymbol{e}_a\}$ be orthonormal, that is

$$\boldsymbol{g}(\boldsymbol{e}_a, \boldsymbol{e}_b) = e_a{}^{\mu}\, e_b{}^{\nu}\, g_{\mu\nu} = \delta_{ab}\,, \tag{3.40}$$

whose inverse relation is given by

$$g_{\mu\nu} = e^a{}_{\mu}\, e^b{}_{\nu}\, \delta_{ab}\,. \tag{3.41}$$

where $e^a{}_{\mu}$ is the inverse of $e_a{}^{\mu}$ where $e^a{}_{\mu}$ is the inverse of $e_a{}^{\mu}$ when we see it as a $n \times n$ matrix. For example, if V^a are the components of the a vector field $\boldsymbol{V}$ with to respect to non-coordinate frame $\{\boldsymbol{e}_a\}$, it follows that $V^a = \boldsymbol{e}^a(\boldsymbol{V}) = e^a{}_{\mu} V^{\mu}$, where V^{μ} are the components on the frame $\{\partial_\mu\}$. Associated to non-coordinate vector field frame $\{\boldsymbol{e}_a\}$ is the so-called dual frame of 1-forms $\{\boldsymbol{e}^a\}$, defined to be such that its action on the frame $\boldsymbol{e}_a$ is: $\boldsymbol{e}^a(\boldsymbol{e}_b) = \delta^a{}_b$, the elements of such frame are 1-forms fields given by

$$\boldsymbol{e}^a = e^a{}_{\mu}\, dx^{\mu}\,. \tag{3.42}$$

The components of a 1-form field $\boldsymbol{\omega}$ with to respect this frame are $\omega_a = \boldsymbol{\omega}(\boldsymbol{e}_a) = e_a{}^{\mu}\omega_\mu$. The convenient choose of the non-coordinate frame $\{\boldsymbol{e}^a\}$ enables us work with the metric on a useful way. Using (3.41), it immediate see that

$$\boldsymbol{g} = g_{\mu\nu}\, dx^{\mu} \otimes dx^{\nu} = \delta_{ab}\, \boldsymbol{e}^a \otimes \boldsymbol{e}^b\,. \tag{3.43}$$

The frames $\{\boldsymbol{e}_a\}$ and its dual $\{\boldsymbol{e}^a\}$ constitute what are called non-coordinate basis. It is worth note that, on a coordinate frame we have $d(dx^a) = 0$, while that on a non-coordinate frame we have $d(d\boldsymbol{e}^a) \neq 0$. Once fixed the frame $\{\boldsymbol{e}_a\}$, we can define a set of n^2 spin-connection 1-forms $\boldsymbol{\omega}^a{}_b$ by

$$V^{\mu}\nabla_{\mu}\boldsymbol{e}^a = -\boldsymbol{\omega}^a{}_b(\boldsymbol{V})\boldsymbol{e}^b \quad \forall \quad \boldsymbol{V} \in TM\,, \tag{3.44}$$

where $\boldsymbol{\omega}^c{}_b = \omega_{ab}{}^c\, \boldsymbol{e}^a$. Using (3.39)and (3.30), after some algebra, we can prove that spin connection 1-forms are related with the torsion and curvature by the following identities [20, 21, 22]

$$\mathcal{T}^a = d\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b \quad , \quad \mathcal{R}^a{}_b = d\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b\,. \tag{3.45}$$

The object $\mathcal{T}^a = \frac{1}{2} T^a{}_{bc}\, \boldsymbol{e}^b \wedge \boldsymbol{e}^c$ is called the torsion 2-form of the connection ∇ and $\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \boldsymbol{e}^c \wedge \boldsymbol{e}^d$ the curvature 2-form of the connection ∇, where $T^a{}_{bc}$ and $R^a{}_{bcd}$ are the components of the torsion and Riemann tensors with to respect to the non-coordinate frame $\{\boldsymbol{e}^a\}$. We should keep in mind that, for example, $\mathcal{R}^a{}_b$ represents the entire Riemann tensor, with Greek indices suppressed. These equations are known as the Cartan's structure equations. The coefficients of the spin connection 1-forms with all indices down, $\omega_{abc} = \omega_{ab}{}^d g_{dc}$, are called Ricci rotation coefficients. If $\{\boldsymbol{e}_a\}$ is orthonormal then the components of the metric g_{ab} in this frame are constants. We can choose a connection according to which the metric is covariantly constant tensor, $\nabla_\sigma g_{\mu\nu} = 0$. In particular, in considering this, we can see that coefficients of the spin connection 1-form with all low indices are antisymmetric in their two last indices, $\omega_{abc} = -\omega_{acb}$. In terms of spin connection 1-forms $\boldsymbol{\omega}_{ab} = g_{ac}\,\boldsymbol{\omega}^c{}_b$ this is expressed as

$$\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}\,, \tag{3.46}$$

which is the antisymmetry condition. Moreover, and foremost, if the connection satisfies $\nabla_\sigma g_{\mu\nu} = 0$ and the torsion-free condition, which is expressed as

$$d\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b = 0\,, \tag{3.47}$$

the connection is uniquely determined. Note that this condition implies immediately the symmetry of the Christoffel symbols. Actually, one can prove that the Levi-Civita connection is the unique among the connections which satisfies the two properties simultaneously. In general relativity, we always assume that the torsion vanishes. Cartan's structure equations provides a fruitful and quicker way to compute the Riemann tensor of a manifold, see [20, 28, 21] for some applications or wait for the next section in which we make an interesting application.

In physics, the relevance of these equations stems from description of the fundamental interactions of nature as a gauge theory, more precisely Yang-Mills theory. This theory is a generalization, with non-abelian symmetry group, of the eletromagnetism which is, in turn, an abelian gauge theory under the group $U(1)$. Moreover, the gauge theory is a type of field theory in which the Lagrangian is invariant under a continuous group of local transformations. This means that the physics should not depend on how we describe it, which in accordance with the principles of general relativity [4, 21].

3.4 Symmetries, Killing Vectors and Hidden Symmetries

A manifold $(M, \boldsymbol{g})$ is said to possess a symmetry if its geometry is invariant under any transformation that maps M to itself. A map which realizes this transformation, denoted by ϕ, is called a diffeomorphism. Let ϕ_τ be a family of such maps on M, generated by a vector field $\boldsymbol{V}$ whose integral curves are $dx^\mu(\tau)/d\tau = V^\mu$,

$$\begin{array}{rccl} \phi_\tau : & M & \rightarrow & M \\ & x^\mu & \mapsto & \phi_\tau(x^\mu) \equiv x^\mu(\tau)\,. \end{array} \tag{3.48}$$

The set of such maps on M forms an abelian group of M if for each $\tau \in \mathbb{R}$ we have a diffeomorphism ϕ_τ satisfying

$$\phi_\tau \circ \phi_s = \phi_{\tau+s} \quad , \quad \phi_0 \text{ is the identity}\,. \tag{3.49}$$

A diffeomorphism on the 2-sphere, for example, is given by $\phi_\tau(\theta, \phi) = (\theta, \phi + \tau)$. This map provides a natural way of comparing tensors at different points on a manifold. For example, any tensor at initial point τ_0 is dragged along the curve $x^\mu(\tau)$ by this map. The set theses curves fill[1] the manifold M. If the metric tensor is the same from one point to another on a manifold $(M, \boldsymbol{g})$ irrespective of the value of the coordinate x^μ, it means that the geometry of $(M, \boldsymbol{g})$ does not change. When this happens we say that the diffeomorphism ϕ_τ generated by a vector field $\boldsymbol{V}$ is a symmetry of this tensor field. Isometries are diffeomorphisms that preserve the metric. But, given an isometry, who are its generators? Let us find it now. Let ϵ be an infinitesimal parameter and $\boldsymbol{K}$ a vector field. Suppose the map ϕ_ϵ drags the point x^μ up to the point $\phi_\epsilon(x^\mu) = x'^\mu$ in the direction of $\boldsymbol{K}$ by a parameter ϵ along of the integral curves of $\boldsymbol{K}$. Then,

$$x'^\mu = x^\mu + \epsilon K^\mu(x) + O(\epsilon^2) \Rightarrow \frac{\partial x'^\mu}{\partial x^\rho} = \delta^\mu_\rho + \epsilon\, \partial_\rho K^\mu + O(\epsilon^2)\,. \tag{3.50}$$

Now, expanding the metric $g_{\mu\nu}(x')$ in Taylor series, we have

$$g_{\mu\nu}(x') = g_{\mu\nu}(x) + \epsilon K^\sigma \partial_\sigma g_{\mu\nu} + O(\epsilon^2)\,, \tag{3.51}$$

and using (3.50) and (3.11), it is easy to prove that the components of the metrics $g'_{\mu\nu}$ and $g_{\mu\nu}$ are related by the following equation:

$$g_{\mu\nu}(x') = g_{\mu\nu}(x) + \epsilon\, \pounds_{\boldsymbol{K}}\, g_{\mu\nu} + O(\epsilon^2)\,, \tag{3.52}$$

where

$$\pounds_{\boldsymbol{K}}\, g_{\mu\nu} \equiv K^\sigma \partial_\sigma g_{\mu\nu} + (\partial_\mu K^\sigma) g_{\sigma\nu} + (\partial_\nu K^\sigma) g_{\mu\sigma}\,. \tag{3.53}$$

Suppose now that the manifold $(M, \boldsymbol{g})$ is symmetric in the direction of the vector field $\boldsymbol{K} = \partial_\sigma$ which in component notation is given by $K^\mu = (\partial_\sigma)^\mu = \delta^\mu_\sigma$. This means that the components of a metric $g_{\mu\nu}$ are invariant by the transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon K^\mu(x)$. Then, from the equation (3.51) we have that:

$$g_{\mu\nu}(x') = g_{\mu\nu}(x) \Rightarrow \pounds_{\boldsymbol{K}}\, g_{\mu\nu} = 0\,. \tag{3.54}$$

[1]Formally, this is the idea of a congruence. A congruence is set of curves that fill the manifold M, or some part of it, without intersecting and it is such that each point in the region of M is on one and only one curve [16, 21].

The operator $\pounds_{\boldsymbol{K}}$ is the so-called **Lie derivative** along of $\boldsymbol{K}$. It is worth note that, in this case, the Lie derivative is simply the partial derivative, $\partial_\sigma g_{\mu\nu} = 0$. A **Killing vector field** is defined to be a vector field $\boldsymbol{K}$ along of which the Lie derivative of the metric vanishes. We say that the vector $\boldsymbol{K}$ generates the isometry, *i.e.*, the transformation under which the geometry is invariant is expressed infinitesimally as a motion in the direction of $\boldsymbol{K}$. Even though the independence of the metric components on one or more coordinates implies the existence of isometries, the converse does not necessarily hold. Sometimes the existence of isometries is not trivial. For instance, the Minkowski manifold $(\mathbb{R}^4, \boldsymbol{\eta})$, has 10 independent Killing vector fields, although only 4 symmetries are obvious from the usual expression of this metric in Cartesian coordinates. In the next section this will be best illustrated with an example. In this sense, the Lie derivative characterizes the symmetries of a manifold without explicitly use coordinates. Moreover, we can rewrite it in a convenient form. First, note that the operator $\pounds_{\boldsymbol{K}}$ has the remarkable property that when acting on a tensor it yields another tensor. Its action on a general tensor is given by [20, 16]

$$\begin{aligned}
\pounds_{\boldsymbol{K}} T^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q} &= K^\sigma \partial_\sigma T^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q} \\
&- (\partial_\sigma K^{\mu_1})\, T^{\sigma\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q} \\
&- (\partial_\sigma K^{\mu_2})\, T^{\mu_1\sigma\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q} - \cdots \\
&+ (\partial_{\nu_1} K^{\sigma})\, T^{\mu_1\mu_2\ldots\mu_p}{}_{\sigma\nu_2\ldots\nu_q} \\
&+ (\partial_{\nu_2} K^{\sigma})\, T^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\sigma\ldots\nu_q} + \cdots .
\end{aligned} \tag{3.55}$$

Note also that although we have used the partial derivative, we could have used the covariant derivative and the result would be the same, since all of the terms that would involve Christoffel symbol would cancel because of its the symmetry, $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$. Therefore,

$$\pounds_{\boldsymbol{K}}\, g_{\mu\nu} = K^\sigma \nabla_\sigma g_{\mu\nu} + \nabla_\mu K_\nu + \nabla_\nu K_\mu \,, \tag{3.56}$$

and using that the metric is covariantly constant, we obtain that:

$$\pounds_{\boldsymbol{K}}\, g_{\mu\nu} = 0 \Rightarrow \nabla_\mu K_\nu + \nabla_\nu K_\mu = \nabla_{(\mu} K_{\nu)} = 0 \,. \tag{3.57}$$

The above equation is the so-called Killing equation and the vector fields $\boldsymbol{K}$ that satisfy it are known as Killing vector fields.

The Noether theorem is one important theorem in physics which states that continuous symmetries are associated to conserved charges. So, since Killing vectors are generators of symmetries of the space $(M, \boldsymbol{g})$, we can use it to construct scalars that are conserved along geodesic curves. For example, if $\boldsymbol{K}$ is Killing vector field and $x(\tau)$, with τ an affine parameter, a geodesic curve whose tangent vector field is $\boldsymbol{p}$, then following scalar is conserved along such geodesic.

$$c = p_\mu K^\mu = p^\mu K_\mu \quad , \quad p^\mu = \frac{dx^\mu(\tau)}{d\tau} \,. \tag{3.58}$$

Indeed, any vector K^μ that satisfies $\nabla_{(\mu}K_{\nu)} = 0$ implies that

$$p^\mu \nabla_\mu (K_\nu p^\nu) = p^\mu p^\nu \nabla_{(\mu}K_{\nu)} = 0\,.$$

This can be understood physically. In the classical dynamics of particles, suppose that the potential is independent of the Cartesian coordinate x, a free particle will not feel any forces in this direction, then it known that the linear momentum is conserved along the direction x. Since a Killing vector field generates an isometry, this shows that symmetry transformations of the metric give rise to conservation laws.

It is immediate to conclude that a linear combination with constant coefficients of two Killing vectors is again a Killing vector, that is, the Killing vectors form a linear space. But, given two Killing vector fields $\boldsymbol{K}$ and $\boldsymbol{H}$, is the final result of the operation $[\boldsymbol{K}, \boldsymbol{H}] = \boldsymbol{K}\,\boldsymbol{H} - \boldsymbol{H}\,\boldsymbol{K}$ a Killing vector? The answer is yes. In particular, one can prove that this commutator is also a vector field given by

$$[\boldsymbol{K}, \boldsymbol{H}]^\mu = K^\nu \partial_\nu H^\mu - H^\nu \partial_\nu K^\mu\,. \tag{3.59}$$

One can easily prove that acting on functions and fields, the Lie derivative along $[\boldsymbol{K}, \boldsymbol{H}]$ satisfy the equation $\pounds_{[\boldsymbol{K},\boldsymbol{H}]} = [\pounds_{\boldsymbol{K}}, \pounds_{\boldsymbol{H}}] = \pounds_{\boldsymbol{K}}\pounds_{\boldsymbol{H}} - \pounds_{\boldsymbol{H}}\pounds_{\boldsymbol{K}}$. Applying this relation to the metric one obtains $\pounds_{[\boldsymbol{K},\boldsymbol{H}]}\, g_{\mu\nu} = 0$. Thus, the commutator $[\boldsymbol{K}, \boldsymbol{H}]$ is again a Killing vector. This commutator is sometimes called the Lie bracket. Since the commutator of Lie derivatives is the Lie derivative with respect to a Lie bracket of vector fields, in the neighbourhood of any point in M the Killing vector fields form a Lie algebra under the Lie bracket. Let $\{K_i\}$ $(i = 1, 2, \ldots, r)$ be a basis in the linear space of the Killing vectors, then the algebra Lie is completely characterized in this basis by its structure constants C^k_{ij} defined by the following relation

$$[\boldsymbol{K}_i, \boldsymbol{K}_j] = C_{ij}{}^k \boldsymbol{K}_k\,. \tag{3.60}$$

By the Frobenius theorem, is well-known that a distribution generated by the vector fields $\{\boldsymbol{K}_i\}$ is integrable if, and only if, there exists a set of structure constants $C_{ij}{}^k$ such that the equation (3.60) is satisfied. In particular, when all the structure constants vanish, the lie algebra is said to be Abelian. Thus, the set of Killing vectors fields $\{K_i\}$ generates a r-dimensional integrable distribution, that is, they span a r-dimensional vector subspace $\Delta_p \subset T_pM$ on every point $p \in M$ and there exist a smooth family of submanifolds of $(M, \boldsymbol{g})$ such that the tangent spaces of these submanifolds are Δ_p.

3.4.1 Maximally Symmetric Manifolds

A manifold M is said to be homogeneous and isotropic if it is invariant under any translation along a coordinate and if it is invariant under any rotation of a coordinate into another coordinate, respectively. In general, a manifold will admit no symmetry, and hence possesses no Killing vectors. Furthermore, there is a maximum number of linearly independent Killing fields that can exist for any metric on M. The manifolds with highest degree of symmetry are said to be homogeneous and isotropic manifolds. A **maximally symmetric manifold** is a manifold possessing the maximum number of **isometries**, which is $n(n+1)/2$ in n dimensions. In particular, as we will see, these manifolds has constant curvature and manifolds with constant curvature can be shown to be conformally flat. Indeed, in the case of Euclidean signature, one can find a coordinate system $\{x^\mu\}$ in which the line element is:

$$ds^2 = \frac{1}{\Omega^2}[(dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2], \tag{3.61}$$

with a yet unknown arbitrary function $\Omega = \Omega(x^\mu)$. Often, it is convenient to use a non-coordinate frame. Identifying the coefficients of the linear combination (3.42) with $\boldsymbol{e}^a{}_\mu = \frac{1}{\Omega}\delta^a{}_\mu$, it leads us to introduce the following dual frame

$$\boldsymbol{e}^a = \frac{1}{\Omega}\, dx^a, \tag{3.62}$$

from which we find

$$d\boldsymbol{e}^a = -\frac{\partial_b\Omega}{\Omega^2}\, dx^b \wedge dx^a = (\partial_b\Omega)\, \boldsymbol{e}^a \wedge \boldsymbol{e}^b. \tag{3.63}$$

In this frame, the line element can be written as

$$ds^2 = (\boldsymbol{e}^1)^2 + (\boldsymbol{e}^2)^2 + \ldots + (\boldsymbol{e}^n)^2 = \delta_{ab}\, \boldsymbol{e}^a\, \boldsymbol{e}^b. \tag{3.64}$$

Since $\delta_{ab} = \boldsymbol{g}(\boldsymbol{e}_a, \boldsymbol{e}_b)$, we do not need to distinguish between raised and lowered indices. Ordinarily, Cartan's structure equations surpasses in efficiency every other known method for calculating the curvature of a manifold. As an exercise let us compute the curvature in terms of Ω for this space using the Cartan equations and the fact that maximally symmetric solutions are regarded as solutions of $R_{ab} = \Lambda\, g_{ab}$ with $\Lambda = R/n$. Manifolds whose Ricci tensor satisfies the latter equation are called an Einstein manifolds. Let us find the function Ω which provides a solution for this latter equation. We will assume that the torsion vanishes. Once chosen the metric and the frame, we should calculate spin connection 1-forms $\boldsymbol{\omega}^a{}_b$. Using (3.63), the Cartan's first equation (3.45) implies the spin connection 1-forms

$$\boldsymbol{\omega}_{ab} = (\partial_a\Omega)\boldsymbol{e}_b - (\partial_b\Omega)\boldsymbol{e}_a. \tag{3.65}$$

According to Cartan's second structure equation, the curvature 2-forms are

$$\begin{aligned}\mathcal{R}_{ab} &= d\boldsymbol{\omega}_{ab} + \boldsymbol{\omega}_{ac} \wedge \boldsymbol{\omega}^c{}_b \\ &= \Omega\,(\partial_a\partial_c\Omega\, \boldsymbol{e}^c \wedge \boldsymbol{e}_b - \partial_b\partial_c\Omega\, \boldsymbol{e}^c \wedge \boldsymbol{e}_a) - (\partial_c\Omega\partial^c\Omega)\, \boldsymbol{e}_a \wedge \boldsymbol{e}_b,\end{aligned} \tag{3.66}$$

This implies that the Riemann tensor is given by

$$R_{abcd} = 2\Omega\,(\partial_a\partial_{[c}\Omega\delta_{d]b} + \partial_b\partial_{[d}\Omega\delta_{c]b}) - 2(\partial\Omega)^2\,\delta_{a[c}\delta_{d]b}. \tag{3.67}$$

where $(\partial\Omega)^2 \equiv \partial^c\Omega\partial_c\Omega$. Contracting the first and third indices we obtain the Ricci tensor,

$$R_{ab} \,=\, (n-2)\Omega\,\partial_a\partial_b\Omega \,+\, \delta_{ab}[\Omega\,\Delta\Omega \,-\, (n-1)\,(\partial\Omega)^2]\,. \tag{3.68}$$

Here $\Delta\Omega \equiv \partial^c\partial_c\Omega$. Hence, the trace of the Ricci tensor yields the curvature scalar,

$$R \,=\, 2(n-1)\,\Omega\,\Delta\Omega \,-\, n(n-1)(\partial\Omega)^2\,. \tag{3.69}$$

Now, using that our manifold is an Einstein manifold, $R_{ab} \,=\, \Lambda\,\delta_{ab}$, this immediately implies

$$\partial_a\partial_b\Omega \,=\, 0 \quad (a \neq b)\,, \tag{3.70}$$

thus, Ω must be a sum of functions depending just on the coordinates x^a

$$\Omega \,=\, \Omega_1(x^1) \,+\, \Omega_2(x^2) \,+\ldots+\, \Omega_n(x^n)\,, \tag{3.71}$$

because otherwise the mixed derivatives could not vanish. However, for $a = b$ we should solve the following equation

$$(n-2)\Omega\,\partial_a\partial_a\Omega \,=\, \Omega\,\Delta\Omega \,-\, (n-1)\,(\partial\Omega)^2 \,+\, \Lambda\,. \tag{3.72}$$

Since the right-hand side does not depend on index a, all the second derivative $\partial_a\partial_a\Omega$ must be equal and constant, which we choose conveniently as

$$\partial_a\partial_a\Omega \,=\, 2\kappa\,, \tag{3.73}$$

and Ω must be quadratic in x^a with a coefficient of $(x^a)^2$ which is independent of x^a. Therefore, we can write

$$\Omega \,=\, c \,+\, \kappa\, r^2\,, \tag{3.74}$$

where $r^2 \,=\, [(x^1)^2 \,+\, (x^2)^2 \,+\ldots+\, (x^n)^2] \,=\, x^a x_a$. Note that if the linear term is non-null, it can be made zero by translating the coordinate origin, and a constant factor on Ω is irrelevant because it simply scales the coordinates, hence we can fix $c = 1$ without loss of generality. By means of the equation (3.74), we have that $(\partial\Omega)^2 \,=\, 4\kappa^2 r^2$ and $\Delta\Omega \,=\, 2n\kappa$. Then, inserting these latter identities into (3.72), we find that the constant κ must be equal to

$$\Lambda \,=\, 4\kappa(n-1) \quad\Rightarrow\quad \kappa \,=\, \frac{R}{4n(n-1)}\,, \tag{3.75}$$

where we use that $\Lambda \,=\, R/n$. The constant κ is the so-called the **curvature parameter**. Finally, an important property of maximally symmetric manifolds is that it has constant curvature. Moreover, by substitution of (3.74) on the equation (3.67) we easily find that:

$$R_{abcd} \,=\, 4\kappa\,(\delta_{ac}\delta_{db} \,-\, \delta_{ad}\delta_{cb})\,. \tag{3.76}$$

Conversely, if the Riemann tensor satisfies this condition the manifold $(M, \boldsymbol{g})$ will be maximally symmetric. In general, Maximally symmetric manifolds of dimension n can be classified by their signature, their curvature parameter and discrete information related to the global topology. But locally, if we ignore questions about global topology, maximally symmetric manifolds with Euclidean signature are fully specified by their curvature parameter of the following form:

$$M = \begin{cases} \mathbb{S}^n & , \quad \kappa > 0 \\ \mathbb{R}^n & , \quad \kappa = 0 \;, \\ \mathbb{H}^n & , \quad \kappa < 0 \end{cases} \tag{3.77}$$

where $\mathbb{S}^n$ is the n-sphere, $\mathbb{R}^n$ the Euclidean space and $\mathbb{H}^n$ the hyperbolic space. Note that, when considering Lorentzian signature, the maximally symmetric manifold with $\kappa = 0$ is the Minkowski manifold. The metric of maximally symmetric spaces with other signatures, like de Sitter and Anti-de Sitter spacetimes, can be obtained from (3.61) by means of analytical continuations of the form $x^a \to i x^a$ [29]. Before proceeding, let us clarify the previous concepts in this section with an example.

Example 4: The Schwarzschild-AdS black hole in arbitrary dimensions.

In Lorentzian signature, the anti-de Sitter (AdS) manifold is a solution with $\Lambda < 0$, of the equation $R_{ab} = \Lambda\, g_{ab}$. We want to find the solution in arbitrary dimension of the latter equation for the Schwarzschild-AdS space whose line element can be written as follows:

$$\begin{aligned} ds^2 &= -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + \frac{r^2}{(1 + \kappa r^2)^2}[(dx^1)^2 + (dx^2)^2 + \ldots + (dx^{n-1})^2] \\ &= -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2\, \delta_{ab}\, \hat{\boldsymbol{e}}^a \hat{\boldsymbol{e}}^b \,, \end{aligned} \tag{3.78}$$

where $\hat{\boldsymbol{e}}^a$ $(a = 1, 2, \ldots, n-1)$ is an orthonormal frame on the $(n-1)$-sphere. Note that all metric components are independent of the coordinate t, therefore this metric possesses the Killing vector $\boldsymbol{K} = \partial_t$. This is the general form of a metric describing a static spherically symmetric spacetime geometry. Here follows the stepwise algorithm we use to determine the components of the Ricci tensor.

1. A suitable orthonormal frame for this a space is given by

$$\boldsymbol{e}^t = e^{\alpha(r)}\, dt \quad , \quad \boldsymbol{e}^r = e^{\beta(r)}\, dr \quad , \quad \boldsymbol{e}^a = r\, \hat{\boldsymbol{e}}^a \,. \tag{3.79}$$

In this frame, the line element is given by

$$ds^2 = -(\boldsymbol{e}^t)^2 + (\boldsymbol{e}^r)^2 + \delta_{ab}\, \boldsymbol{e}^a \boldsymbol{e}^b = \eta_{\alpha\beta}\; \boldsymbol{e}^\alpha \boldsymbol{e}^\beta \qquad (\alpha = t, r, a)\,, \tag{3.80}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, \ldots, 1)$ is the Minkowski manifold in $n+1$ dimensions.

2. Computing the spin connection 1-forms by applying Cartan's first structure equation, namely

$$d\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b \,=\, 0\,.$$

Taking the exterior derivative of $\boldsymbol{e}^t$, it is simple to show that:

$$d\boldsymbol{e}^t \,=\, -e^{\beta(r)}\alpha'(r)\,\boldsymbol{e}^t \wedge \boldsymbol{e}^r \;\Rightarrow\; \boldsymbol{\omega}^t{}_r \,=\, -e^{\beta(r)}\alpha'(r)\,\boldsymbol{e}^t\,,$$

where $\alpha'(r)$ stands for the derivative of $\alpha(r)$ with respect to its variable r. Continuing in the same manner, all the non-vanishing spin connection 1-forms are the following:

$$\boldsymbol{\omega}^t{}_r \,=\, -e^{\beta(r)}\alpha'(r)\,\boldsymbol{e}^t \quad , \quad \boldsymbol{\omega}^a{}_r \,=\, -\frac{e^{\beta(r)}}{r}\,\boldsymbol{e}^a \quad , \quad \boldsymbol{\omega}^a{}_b \,=\, \hat{\boldsymbol{\omega}}^a{}_{\hat{b}}\,. \tag{3.81}$$

3. Determining the curvature 2-forms by use of Cartan's second structure equation

$$\mathcal{R}^a{}_b \,=\, d\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b\,.$$

Let us do the less trivial case. From (3.81), we have that $\boldsymbol{\omega}^a{}_b \,=\, \hat{\boldsymbol{\omega}}^a{}_{\hat{b}}$, then

$$\begin{aligned}\mathcal{R}^a{}_b \;&=\; d\hat{\boldsymbol{\omega}}^a{}_b \,+\, \hat{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{\omega}}^b{}_c \,+\, \boldsymbol{\omega}^a{}_t \wedge \boldsymbol{\omega}^t{}_b \,+\, \boldsymbol{\omega}^a{}_r \wedge \boldsymbol{\omega}^r{}_b \\ &=\; d\hat{\boldsymbol{\omega}}^a{}_b \,+\, \hat{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{\omega}}^b{}_c \,+\, \frac{e^{-2\beta(r)}}{r^2}\,\boldsymbol{e}^a \wedge \boldsymbol{e}^b \\ &=\; \hat{\mathcal{R}}^a{}_b \,-\, \frac{e^{-2\beta(r)}}{r^2}\,\boldsymbol{e}^a \wedge \boldsymbol{e}^b\,,\end{aligned} \tag{3.82}$$

where $\hat{\mathcal{R}}^a{}_b$ is the curvature 2-forms derived from the connection 1-forms $\hat{\boldsymbol{\omega}}_{ab}$ with respect to the frame of 1-form $\{\hat{\boldsymbol{e}}^a\}$ defined on the $(n-1)$-sphere.

4. By applying the the relation

$$\mathcal{R}^a{}_b \,=\, \frac{1}{2}\,R^a{}_{bcd}\,\boldsymbol{e}^c \wedge \boldsymbol{e}^d$$

and using the skew-symmetry of the exterior product, we find that:

$$R^a{}_{bcd} \,=\, \frac{\hat{R}^a{}_{bcd}}{r^2} \,-\, \frac{e^{-2\beta(r)}}{r^2}\,(\delta_{ac}\delta_{db} \,-\, \delta_{ad}\delta_{cb})\,,$$

where $\hat{R}^a{}_{bcd}$ is the Riemann tensor with respect to the frame of 1-forms $\{\hat{\boldsymbol{e}}^a\}$. Now, using the fact that the sphere is a maximally symmetric manifold and therefore its Riemann tensor satisfies the equation

$$R_{abcd} \,=\, 4\kappa\,(\delta_{ac}\delta_{db} \,-\, \delta_{ad}\delta_{cb})\,,$$

we get the components

$$R_{abcd} = \frac{1}{r^2}\left(4\kappa - e^{-2\beta(r)}\right)\left(\delta_{ac}\delta_{db} - \delta_{ad}\delta_{cb}\right).$$

Here, we must choose the curvature parameter as being $\kappa = 1/4$, since this choice means the term of the metric $\delta_{ab}\,\hat{\boldsymbol{e}}^a\hat{\boldsymbol{e}}^b$ is an unit $(n-1)$sphere. Thus, the Riemann tensor is given by

$$\begin{aligned} R_{abcd} &= \frac{1}{r^2}\left(1 - e^{-2\beta(r)}\right)\left(\delta_{ac}\delta_{db} - \delta_{ad}\delta_{cb}\right) \\ \Rightarrow R^a{}_{bab} = \delta^{ac}R^a{}_{bcb} &= \frac{(n-2)}{r^2}\left(1 - e^{-2\beta(r)}\right), \end{aligned} \tag{3.83}$$

where for the index b Einstein's convention is not being used. Analogously, it is straightforward obtain the other components of the Riemann tensor that are non-vanishing, they are,

$$\begin{aligned} R^t{}_{rtr} &= e^{-2\beta}[\alpha''(r) - \beta'(r)\alpha'(r) + \alpha'(r)^2] \quad , \quad R^r{}_{ara} = \frac{e^{-2\beta}\beta'(r)}{r}, \\ R^t{}_{ata} &= -\frac{e^{-2\beta}\alpha'(r)}{r} \quad , \end{aligned} \tag{3.84}$$

where for the index a we do not employ Einstein's convention.

5. Contraction of these components gives the components of the Ricci tensor, $R_{\gamma\delta} = \eta^{\alpha\beta}R_{\alpha\gamma\beta\delta}$, and using the four symmetries of the Riemann curvature tensor, we arrive at the following equations:

$$R_{tt} = R^\alpha{}_{t\alpha t} = e^{-2\beta}\left[\alpha''(r) - \beta'(r)\alpha'(r) + \alpha'(r)^2 + \frac{(n-1)\alpha'(r)}{r}\right] \tag{3.85}$$

$$R_{rr} = R^\alpha{}_{r\alpha r} = -\,e^{-2\beta}\left[\alpha''(r) - \beta'(r)\alpha'(r) + \alpha'(r)^2 - \frac{(n-1)\beta'(r)}{r}\right] \tag{3.86}$$

$$R_{aa} = R^\alpha{}_{a\alpha a} = \left[e^{-2\beta}\frac{\beta'(r) - \alpha'(r)}{r} + \frac{(n-2)}{r^2}\left(1 - e^{-2\beta(r)}\right)\right]. \tag{3.87}$$

6. In the end we use the fact that our spacetime is an Einstein manifold, that is,

$$R_{\alpha\beta} = \Lambda\,\eta_{\alpha\beta} \Rightarrow \begin{cases} R_{tt} = -\Lambda \\ R_{rr} = \Lambda \\ R_{aa} = \Lambda \end{cases}.$$

Using this condition, the equations (3.85) and (3.86) immediately lead to

$$\alpha'(r) = -\beta'(r) \Rightarrow \alpha + \beta = \lambda, \tag{3.88}$$

where λ is a constant. Note that, by choosing a suitable coordinate time $t \to e^{\lambda}t$, we can achieve $\lambda = 0$ and it follows that $\alpha = -\beta$. Inserting the equation (3.88) into (3.87) and noting that

$$[r^{n-2}(1-e^{-2\beta(r)})]' = (n-2)r^{n-3}(1-e^{-2\beta(r)}) + 2\,r^{n-2}\,e^{-2\beta(r)}\beta'(r)\,,$$

it is straightforward to prove that from $R_{aa} = \Lambda$, it follows the relation:

$$e^{-2\beta(r)} = 1 - \frac{C}{r^{n-2}} - \frac{\Lambda\,r^2}{n} = e^{2\alpha(r)}\,, \tag{3.89}$$

where C is an arbitrary constant. Therefore, the metric (3.78) becomes

$$\begin{aligned} ds^2 &= -\left(1 - \frac{C}{r^{n-2}} - \frac{\Lambda\,r^2}{n}\right) dt^2 + \left(1 - \frac{C}{r^{n-2}} - \frac{\Lambda\,r^2}{n}\right)^{-1} dr^2 \\ &+ r^2\,\delta_{ab}\,\hat{\boldsymbol{e}}^a\hat{\boldsymbol{e}}^b\,. \end{aligned} \tag{3.90}$$

This metric, known as the Schwarzschild-AdS metric for $\Lambda < 0$, looks like the Schwarzschild black hole for small r and approaches the AdS space for large r. The Schwarzschild-AdS black hole is the unique static, spherically symmetric solution of the equation $R_{\alpha\beta} = \Lambda\eta_{\alpha\beta}$.

□

3.4.2 Symmetries of Euclidean Space $\mathbb{R}^n$

An example of a space with the highest possible degree of symmetry is $\mathbb{R}^n$ with the flat Euclidean metric. Let us examine its symmetries. Consider the n-dimensional flat space, for which the metric is Euclidean. In cartesian coordinates $\{x^\mu\}$, the metric is given by

$$\delta_{\mu\nu} = \mathrm{diag}(1,1,\ldots,1)\,. \tag{3.91}$$

The Taylor series of a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ near the origin of the coordinate system is a power series of the form

$$f(x) = q + q_\mu x^\mu + \frac{1}{2!}q_{\mu\nu}x^\mu x^\nu + \frac{1}{3!}q_{\mu\nu\sigma}x^\mu x^\nu x^\sigma + \ldots + \frac{1}{n!}q_{\mu_1\ldots\mu_n}x^{\mu_1}\ldots x^{\mu_n}\,,$$

where, for simplicity, q means the function evaluated at $x^\mu = 0$ and $q_{\nu_1\ldots\nu_n}$ its partial derivatives evaluated at the latter point. In the same way, the Taylor series expansion for a vector field $\boldsymbol{K} = K^\mu\partial_\mu$ can be represented near point x as:

$$K_\mu = q_\mu + q_{\mu\nu}x^\nu + \frac{1}{2!}q_{\mu\nu_1\nu_2}x^{\nu_1}x^{\nu_2} + \frac{1}{3!}q_{\mu\nu_1\nu_2\nu_3}x^{\nu_1}x^{\nu_2}x^{\nu_3} + \ldots + \frac{1}{n!}q_{\mu\nu_1\ldots\nu_n}x^{\nu_1}\ldots x^{\nu_n}\,, \tag{3.92}$$

Since that the indices $\nu_1, \nu_2, \ldots, \nu_n$ are completely arbitrary, swapping any two coordiantes, x^{ν_1} and x^{ν_2} for example, the result must be the same. This, in its turn, implies that $q_{\mu\nu_1\ldots\nu_n}$ is symmetric in the indices $\nu_1, \ldots, \nu_n$. The choice of Cartesian coordinates is very convenient, since all Christoffel symbols vanishes and we may replace the covariant derivatives by partial derivatives. Then, the Killing equation is simply

$$\partial_{(\mu} K_{\nu)} \,=\, 0\,.$$

Substituting (3.92) into the above Killing equation we arrive at the following expression

$$q_{(\mu\nu)} \,+\, q_{(\mu\nu)\nu_2} x^{\nu_2} \,+\, q_{(\mu\nu)\nu_2\nu_3} x^{\nu_2} x^{\nu_3} \,+ \ldots +\, q_{(\mu\nu)\nu_2\ldots\nu_n} x^{\nu_2} \ldots x^{\nu_n} \,=\, 0\,,$$

where indices inside round brackets are symmetrized. Since each term in the sum relate different powers in the coordinates and that the functions $x^{\nu}, x^{\nu_2}x^{\nu_3}, \ldots, x^{\nu_2}\ldots x^{\nu_n}$ are linearly independent, it follows that for any x^{μ} each terms of this sum must vanishes,

$$q_{(\mu\nu)} \,=\, 0 \quad , \quad q_{(\mu\nu)\nu_2} \,=\, 0 \quad , \ldots , \quad q_{(\mu\nu)\nu_2\ldots\nu_n} \,=\, 0\,. \tag{3.93}$$

The symmetry of the first two indices in the above equation implies immediately that $q_{\mu\nu}$ must be antisymmetric, $q_{\mu\nu} = -q_{\nu\mu}$. Moreover, it is immediate see that $q_{\mu\nu\nu_2\ldots\nu_n} \,=\, -q_{\nu\mu\nu_2\ldots\nu_n}$. In particular, this latter symmetry and by means of the the symmetry of $q_{\mu\nu\nu_2\ldots\nu_n}$ in the indices $\nu_1, \nu_2, \ldots, \nu_n$ all the derivative of q, except $q_{\mu\nu}$, must vanishes. For example, we know that $q_{\mu\nu\alpha} \,=\, -q_{\nu\mu\alpha}$. However, by the the previous symmetries we obtain the following relation $q_{\nu\mu\alpha} \,=\, -q_{\nu\mu\alpha}$ which is satisfied if and only if $q_{\mu\nu\alpha} = 0$. We can continue with this same procedure until reaching the term $q_{(\mu\nu)\nu_2\ldots\nu_n}$ which, by the same reason, must also vanishes, as was previously stated. We can thus conclude that K_μ must be linear in the coordinates,

$$K_\mu \,=\, q_\mu \,+\, q_{[\mu\nu]}\, x^\nu\,. \tag{3.94}$$

Since that q_μ has n independent components and $q_{[\mu\nu]}$ has $n(n-1)/2$, the equation (3.94) lead us to the conclusion that the Killing vector field has $n(n+1)/2$ independent components. We have, thus, $n(n+1)/2$ independent vector fields, each of the form (3.94) for independent choices of the $n(n+1)/2$ constants q_μ and $q_{[\mu\nu]}$. This implies that the set of all solutions of the Killing equation forms a vector space of dimension $n(n+1)/2$. We can expand the Killing vector field $\boldsymbol{K}$ in terms of its components K_μ on the frame and we arrive at the following equation:

$$\boldsymbol{K} \,=\, K^\mu\,\partial_\mu \,=\, q^\mu\,\partial_\mu \,+\, q^{\mu\nu}\, x_{[\mu}\partial_{\nu]} \,=\, q^\mu \boldsymbol{p}_\mu \,+\, q^{\mu\nu}\,\boldsymbol{L}_{\mu\nu}\,, \tag{3.95}$$

where $\boldsymbol{p}_\mu = \partial_\mu$ is the linear momentum and $\boldsymbol{L}_{\mu\nu} = 2x_{[\mu}\boldsymbol{p}_{\nu]}$ is the angular momentum. The basis $\{\boldsymbol{p}_\mu, \boldsymbol{L}_{\mu\nu}\}$ is the so-called Killing basis. Once we identify a basis, any solution of the Killing equation is therefore a linear combination with constant coefficients of the elements of this basis. The simplest choice of the $n(n+1)/2$

vector fields is obtained taking only the constants q_μ non-null and letting $q_{\mu\nu} = 0$. In this case, the components of $\boldsymbol{p}_\mu$ non-null are n constant vector fields of the form

$$\boldsymbol{K} = \partial_\nu \Rightarrow K^\mu = (\partial_\nu)^\mu = \delta^\mu_\nu \,. \tag{3.96}$$

This represents a unit vector in each of the coordinate directions and are called **translational isometries**. Indeed, from equation (3.50) we see that $x'^\mu = x^\mu + \epsilon\delta^\mu_\nu$. Since they are constant, the integral curves are just the Cartesian coordinate axes, and the metric is indeed independent of each of these coordinates. Now, setting $q_\mu = 0$, we have $n(n-1)/2$ **rotational isometries**. Indeed, choosing one of the $n(n-1)/2$ antisymmetric matrices $q_{\mu\nu}$, $q_{12} = 1/2$ for example, and all the rest zero, it is straightforward to prove that:

$$\boldsymbol{K} = q^{12}\, x_{[1}\boldsymbol{p}_{2]} = x\,\boldsymbol{p}_y - y\,\boldsymbol{p}_x = \boldsymbol{L}_{xy}\,, \tag{3.97}$$

where we identify $\{x^1, x^2\} = \{x, y\}$. One can obtain easily all the $n(n-1)/2$ $\boldsymbol{L}_{[\mu\nu]}$ in analogous ways. In particular, its the algebra is given by the Lie brackets

$$[\boldsymbol{L}_{\mu\nu}, \boldsymbol{L}_{\rho\sigma}] = \delta_{\mu\rho}\boldsymbol{L}_{\nu\sigma} - \delta_{\mu\sigma}\boldsymbol{L}_{\nu\rho} - \delta_{\nu\rho}\boldsymbol{L}_{\mu\sigma} + \delta_{\nu\sigma}\boldsymbol{L}_{\mu\rho}\,. \tag{3.98}$$

The group of all these, translations and rotations, isometries is known as the symmetry group of n-dimensional Euclidean space. So, the most general symmetry transformation which includes both translations and rotations has the form $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, where a^μ is a constant and $\delta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \delta_{\mu\nu}$. Physically, the equation (3.98) is the angular momentum operator algebra. These operators perform rotations in a physical system which do not commute. For instance, an infinitesimal rotation around the 1-axis followed by an infinitesimal rotation around the 2-axis is not the same as rotating around 2 and then 1. The Lie brakets are exactly those of $SO(n)$, the group of rotations in n dimensions. This is no coincidence, of course, but we won't pursue this here. All we need to know here is that a spherically symmetric manifold is one which possesses $n(n+1)/2$ Killing vector fields with the Lie brakets (3.98) [16, 20, 18]. It is possible to establish Lie brackets of this sort for all of the Euclidean Killing fields. Indeed, one can prove that the following relation holds: The group of all these, translations and rotations, isometries is known as the symmetry group of n-dimensional Euclidean space. So, the most general symmetry transformation which includes both translations and rotations has the form $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, where a^μ is a constant and $\delta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \delta_{\mu\nu}$. Physically, the equation (3.98) is the angular momentum operator algebra. These operators perform rotations in a physical system which do not commute. For instance, an infinitesimal rotation around the 1-axis followed by an infinitesimal rotation around the 2-axis is not the same as rotating around 2 and then 1. The Lie brakets are exactly those of $SO(n)$, the group of rotations in n dimensions. This is no coincidence, of course, but we won't pursue this here. All we need to know here is that a spherically symmetric manifold is one which possesses

$n(n+1)/2$ Killing vector fields with the Lie brakets (3.98) [16, 20, 18]. It is possible to establish Lie brackets of this sort for all of the Euclidean Killing fields. Indeed, one can prove that the following relation holds:

$$[\boldsymbol{p}_\mu, \boldsymbol{p}_\nu] = 0 \quad , \quad [\boldsymbol{L}_{\mu\nu}, \boldsymbol{p}_\rho] = \delta_{\mu\rho}\,\boldsymbol{p}_\nu - \delta_{\nu\rho}\,\boldsymbol{p}_\nu \,. \tag{3.99}$$

We therefore find exactly $n(n+1)/2$ isometries in Euclidean space $\mathbb{R}^n$. Actually, this is the maximum number of independent solutions to the Killing equation in a n-dimensional space.

In general relativity, it is well-known that is a manifold admits admits Killing vectors, in general, the corresponding scalars associated with theses vectors can be used to find the geodesics trajectories without integrating the non-linear geodesic equation directly. However, spacetimes with geometries more complicated admits less Killing vectors than dimensions. In this cases, the work to find the geodesic trajectories is much harder. For example, 4-dimensional Kerr spacetime has just two independent Killing vectors and therefore it is not possible to find its geodesics trajectories using only these symmetries. But, there are other tensors associated with symmetries of a manifold. For example, differently from the conserved scalars associated to Killing vectors, which are linear on the momentum, the constant found by Carter in 1968 which enabled him to solve the geodesic equation of the Kerr metric is quadratic on the momentum and it is a consequence of the fact that Kerr metric admits a Killing tensor of order two [26]. This is one example of the so-called the hidden symmetries. We say that a manifold has a hidden symmetries if its geodesic equation possess conserved quantities of higher than first order in momentum. The geometric structure responsible for this kind of symmetry is the Killing tensor [19, 29]. Killing tensors are symmetric generalizations of the concept of Killing vectors. There are also anti-symmetric generalizations of the Killing vectors called Killing-Yano tensors (KY). Theses are more fundamental objects than the Killing tensors. They can be used to construct Killing tensors, since the square of a Killing-Yano is a Killing tensor of order two. For scalars conversed along null geodesic, the tensor responsible for such conservation law is conformal Killing tensor (CKT). One can also find the so-called conformal Killing-Yano tensor (CKY) which is completely anti-symmetric tensor that also lead to conserved quantities along null geodesics.

3.5 Fiber Bundles

In the previous sections it was seen that a manifold is a space which looks locally like $\mathbb{R}^n$, not necessarily globally. Now, we will study certain kind of manifolds which looks locally like a direct product of two manifolds. Such manifolds are called a **fiber bundles**. A particularly interesting manifold is formed by combining a manifold M with all its tangent spaces T_pM. For motivation, let us construct the fiber bundle TM, previously called tangent bundle. It was formed by collecting together all the tangent spaces T_pM from all points of M. The manifold M over which TM is defined is called the base space. Now, suppose that there exist a covering $\{U_i\}$ of M by open sets and diffeomorphism $\{\phi_i\}$ such that $x^\mu = \phi_i(p)$ is the coordinate on U_i to each point $p \in M$ at which one calculates the tangent space T_pM. The union of T_pM over all points of U_i gives

$$TU_i \equiv \bigcup_{p \in U_i} T_pM \,. \tag{3.100}$$

We specify an element q of TU_i by a point $p \in M$ and a vector $\boldsymbol{V} \in T_pM$ decomposed on frame $\{\partial_\mu\}$ at p. Note that U_i can be continuously mapped to an open subset $\phi(U_i)$ of $\mathbb{R}^n$, and T_pM mapped to $\mathbb{R}^n$ and *vice versa*. Then, we can think on TU_i as the direct product $\mathbb{R}^n \times \mathbb{R}^n$, *i.e.*, TU_i is a $2n$-dimensional manifold decomposed into a direct product $U_i \times \mathbb{R}^n$. It means that each point q on TU_i has coordinates (x^μ, V^μ) and contains information of a point $p \in M$ and of a vector $\boldsymbol{V} \in T_pM$. Then, we can introduce a map $\pi : TU_i \to U_i$ called projection such that to any point $q \in TU_i$ the object $\pi(q)$ is a point $p \in U_i$ at which the vector is defined. Now, we can introduce the idea of fibers of a fiber bundle. A fiber above p is the result of the action of the map π^{-1} on point $p \in M$, $\pi^{-1}(p)$. In our case, when p is fixed, it straightforward see that the fibers are exactly the spaces $\pi^{-1}(p) = T_pM$ for each p. Let U_j be a chart as U_i such that $U_i \cap U_j \neq \varnothing$ and $x'^\mu = \phi'_j(p)$ its coordinates. If $\boldsymbol{V} \in T_pM$ is a vector at $p \in U_i \cap U_j$, it is then simple matter to arrive at the following relation

$$\boldsymbol{V} \;=\; V^\mu \partial_\mu \;=\; V'^\mu \partial'_\mu \;\Rightarrow\; V'^\mu \;=\; A^\mu_\nu V^\nu \,, \tag{3.101}$$

i.e., their components are related by a non-singular matrix $A^\mu_\nu \equiv \partial x'^\mu / \partial x^\nu$ of $GL(n, \mathbb{R})$ which act on T_pM by the left with the matrix A^μ_ν. The group $GL(n, \mathbb{R})$ is then called the structure group of TM. Now, a curve in the fiber bundle identifies a particular vector at each point p of M, and so the curve defines a vector field on M. Such a curve which is nowhere parallel to a fiber is called a section σ of TM. Note that it does not make sense to ask for the length, since here we have an example of a manifold on which is not necessary to define a metric. A local section of TU_i is a smooth map $\sigma_i : U_i \to TU_i$ defined on a chart U_i. However, the projection $\pi : TM \to M$ can be defined globally with no reference to local charts, since the equation $\pi(q) = p$ does not depend on a special coordinate chosen. Therefore, we define a section σ of TM as a map $\sigma : M \to TM$ such that $\pi \circ \sigma = e$, where e is the identity on M. The set of sections on M is denoted by $\Gamma(TM)$. Consider now two coordinate neighbourhood $(U_i, \phi_i(p))$ and $(U_j, \phi_j(p))$. The transition function $t_{ij} \in U_i \cap U_j : T_pM \to T_pM$ is given by $t_{ij}(p) = (\phi_i^{-1} \circ \phi_j)(p)$. Since that $\phi_i^{-1} \circ \phi_j$ is a diffeomorphism its derivative is invertible, $t_{ij} \in U_i \cap U_i \in GL(n, \mathbb{R})$. To summarize all this informations, one says that TM is a fiber bundle above M, with fibre T_pM and structure group $GL(n, \mathbb{R})$.

The tangent bundle is an example of a more general framework called a fibre bundle. A fiber bundle E over a smooth manifold M, with fibre F and structure group G, is a set (E, M, F, G, π) such that

(i) E is a smooth manifold called the total space.

(ii) M is a smooth manifold called the base space.

(iii) F is a smooth manifold called the fibre.

(iv) G is a Lie group acting smoothly on F on the left, called the struture group.

(v) π is a smooth surjection[2] from E to M, called the projection. The set $\pi^{-1}(p)$, where $p \in M$, is called the fibre over p, and is denoted by F_p.

(vi) There exist a covering $\{U_i\}$ of M by open sets, and diffeomorphisms $\{\phi_i\}$ mapping $U_i \times F$ to $\pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. The map ϕ_i is called the local trivialization since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$.

(vii) If we write $\phi_i(p, f) = \phi_{i,p}(f)$, the map $\phi_{i,p}(f) : F \to F_p$ is a diffeomorphism. On $U_i \cap U_j \neq \varnothing$, we require that $t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ be an element of G. Then ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ as

$$\phi_j(p, f) \;=\; \phi_i(p, t_{ij}(p) f)\,.$$

The maps t_{ij} are called the transition functions.

We need to clarify several points about the transition functions. For convenience, we will often use a shorthand notation $\pi : E \to M$ to denote a fibre bundle. So, let $\pi : E \to M$ be a fibre bundle and U_i a chart of the base space M. The local trivialization $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$ is a diffeomorphism. Now, let us take a chart U_j such that $U_i \cap U_j \neq \varnothing$. Then, we have two maps ϕ_i and ϕ_j on $U_i \cap U_j$. Let $q \in E$ such $p = \pi(q) \in U_i \cap U_i$. Each local trivialization ϕ_i^{-1} and ϕ_j^{-1} carries the point q to different elements of F:

$$\phi_i^{-1}(q) \;=\; (p, f_i) \quad , \quad \phi_j^{-1}(q) \;=\; (p, f_j)\,. \tag{3.102}$$

The central role of the transition function $t_{ij} : U_i \cap U_j \to G$ consists on establishing a relation between these two elements: $f_i = t_{ij}(p) f_j$. Since $t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p}^{-1}$ one can prove that the following consistency conditions are satisfied:

$$\begin{array}{rcll} t_{ii}(p) & = & e & (p \in U_i) \\ t_{ij}(p) t_{ji}(p) & = & e & (p \in U_i \cap U_j) \\ t_{ij}(p) t_{jk}(p) t_{ki}(p) & = & e & (p \in U_i \cap U_j \cap U_k) \end{array} \quad , \tag{3.103}$$

[2] A map f from a set X to a set Y is said to be a surjection, if every element y in Y has a corresponding element x in X such that $f(x) = y$.

where $e : E \to E$ is the identity element.

The tangent bundle TM constructed in the beginning of this section is a fibre bundle whose fibre is a vector space. These are special types of fibre bundles which are called vector bundles [21, 16]. Another example of a fibre bundle is the bundle of linear frames over M which is closely related to TM, the frame bundle LM. Instead of using the tangent space T_pM as a fibre above a point p of M which was done in the construction of TM, we can take as a fibre at p the set of all the frames of T_pM. Therefore, a point of the fibre F_p above p is a particular frame of T_pM which is given by n linearly independent vectors of T_pM. It means that each point q of LM may be assigned by the coordinates (x^μ, V_a^μ) which contains information of a point $p \in M$ whose coordinates are x^μ and of n^2 components V_a^μ $(a = 1, 2, \ldots, n)$ of the vectors $\boldsymbol{V}_a$ of a frame of T_pM in the coordinate frame $\{\partial_\mu\}$. Since the element V_a^μ form a frame, it is invertible. Hence, V_a^μ is an element $GL(n, \mathbb{R})$, so that the fibre of LM is $GL(n, \mathbb{R})$. Moreover, given any two frames $\{\boldsymbol{V}_a\}$ and $\{\boldsymbol{U}_b\}$ there exists an element of $GL(n, \mathbb{R})$ that relate them and therefore the structure group of LM is $GL(n, \mathbb{R})$. Such a bundle is called a principal fiber bundle, which is a fiber bundle whose fibre F is the same as the structure group G. We establish, thus, that a vector bundle naturally induces a principal fiber bundle over the manifold by employing the same transition functions. In particular, when the frame $\{\boldsymbol{V}_a\}$ is orthonormal, the elements V_a^μ becomes the coefficients of the non-coordinate frame and the structure group reduces to $O(n)$. Furthermore, if the frame $\{\boldsymbol{V}_a\}$ has positive orientation throughout the whole manifold, the corresponding structure group is $SO(n)$.

3.5.1 Spinorial Bundles and Connections

In the chapter 1 we defined spinors algebraically, which arose within the study of Clifford algebra, its Clifford group, and some subgroups of the latter, especially the spin group. It now becomes possible to attack the problem of differentiation of spinor fields. We shall here construct the spinorial bundles which will immediately lead to notion of a spinor fields, the concept of covariant derivative of a spinor fields and at the end we will compute the curvature of the spinorial bundle.

In what follows, the m-dimensional manifold is assumed to have an inner product defined on the bundle TM. Let $\pi : TM \to M$ be a tangent bundle above M, with structure group $O(m)$ or $SO(m)$ when M is orientable and let LM be the frame bundle associated with TM. Given the transition functions $t_{ij} \in SO(m)$ of LM, it is meaningful to consider the set of functions $\tilde{t}_{ij} \in Spin(m)$ defined by

$$\phi(\tilde{t}_{ij}) = t_{ij}\,, \tag{3.104}$$

where ϕ is the double covering $\phi : Spin(m) \to SO(m)$. Since the transition functions t_{ij} satisfy $t_{ij}t_{jk}t_{ki} = e$ and $t_{ii} = e$ and that the mapping ϕ is a representation of the spin group, there are solutions of the equation (3.104) for $\tilde{t}_{ij}$ satisfying $\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = \pm e$ and $\tilde{t}_{ii} = \pm e$ among which we select those $\tilde{t}_{ij}$ satisfying

$$\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} \,=\, e \quad \text{and} \quad \tilde{t}_{ii} \,=\, e\,. \tag{3.105}$$

A **spin structure** on M is defined by the transition functions $\tilde{t}_{ij} \in Spin(m)$ satisfying (3.104) and (3.105). The principal bundle LSM of the spin frames over M is defined as that principal bundle with M as base, with $Spin(m)$ as structure group, and $\tilde{t}_{ij}$ satisfying (3.105). A spinor above $p \in M$ is a linear combination of the elements of the spin frame given by sections of LSM at p. Now, we can introduce of a spinor field in a way completely analogous to definition of a vector field of TM. Once defined the notion of a spinor at a point p, we can construct the **spinorial bundle** SM above p as the bundle which admits as fibre above the point p of M the set of all spinors at p. A spinor field, denoted by $\boldsymbol{\psi}$, over M is then a map the associates to every point p of the manifold M a spinor, it is a section of SM. The set of all sections of SM we denote by $\Gamma(SM)$. It is interesting to note that not all manifolds admit spin structures. For examples of the latter case of spin structures, see [30, 21]. In our case the manifold M is said to admit a spin structure.

From now on we are going to work on a manifold $(M, \boldsymbol{g})$ of dimension $m = 2n$ endowed with a non-degenerate metric $\boldsymbol{g}$. Furthermore, the tangent bundle TM is assumed to be endowed with a torsion-free and metric-compatible derivative, the Levi-Civita connection, and it will be assumed that $(M, \boldsymbol{g})$ admits a spin structure, namely it has a well-defined spinorial bundle.

Consider $\{\boldsymbol{e}_a\}$ a local frame such that the components of the metric in this frame are constants,

$$\boldsymbol{g}(\boldsymbol{e}_a, \boldsymbol{e}_b) \,=\, g_{ab} \quad , \quad \partial_c\, g_{ab} \,=\, 0\,. \tag{3.106}$$

Now, let us introduce a connection $\hat{\nabla}_a$ on the spinorial bundle SM. Let ∇_a be the the Levi-Civita connection of the tangent bundle TM. The action of this connection in the frame vectors $\{\boldsymbol{e}_a\}$ is given by

$$\nabla_a \boldsymbol{e}_b \,=\, \omega_{ab}{}^{c}\boldsymbol{e}_c\,, \tag{3.107}$$

where $\omega_{ab}{}^{c} \equiv \boldsymbol{\omega}^{c}{}_{b}(\boldsymbol{e}_c)$. This connection can be uniquely extended to the spinorial bundle. For such, one might impose for it to satisfy the Leibniz rule with respect to the Clifford action, here denoted by juxtaposition:

$$\hat{\nabla}_a(\boldsymbol{V}\boldsymbol{\psi}) \,=\, (\hat{\nabla}_a\boldsymbol{V})\,\boldsymbol{\psi} \,+\, \boldsymbol{V}(\hat{\nabla}_a\boldsymbol{\psi}) \quad \forall \quad \boldsymbol{V} \in \Gamma(TM)\;, \boldsymbol{\psi} \in \Gamma(SM)\,, \tag{3.108}$$

we refer to it as covariant differentiation of Clifford Products. We must also require it to be compatible with the natural inner products on the spinorial bundle,

$$\hat{\nabla}_a < \boldsymbol{\psi}, \boldsymbol{\phi} > = < \hat{\nabla}_a\boldsymbol{\psi}, \boldsymbol{\phi} > \,+\, < \boldsymbol{\psi}, \hat{\nabla}_a\boldsymbol{\phi} > \quad \forall \quad \boldsymbol{\psi}\,, \boldsymbol{\phi} \in \Gamma(SM)\,. \tag{3.109}$$

Let $\{\boldsymbol{\xi}_A\}$ be a local frame of the spinorial bundle SM, where $A = 1, 2, \ldots, 2^n$ are spinorial indices. The spinors $\boldsymbol{\xi}_A$ give a linear and faithful representation for the Clifford algebra, the so-called spinorial representation. Indeed, we can choose conveniently the spinorial frame so that the Clifford action of the frame $\{\boldsymbol{e}_a\}$ on the spinors $\boldsymbol{\xi}_A$ is constant in a given patch of M,

$$\boldsymbol{e}_a \, \boldsymbol{\xi}_A \equiv (\boldsymbol{e}_a)^B{}_A \boldsymbol{\xi}_B \,, \tag{3.110}$$

where the matrices $(\boldsymbol{e}_a)^B{}_A$ are constants. In the physics literature the matrices $(\boldsymbol{e}_a)^B{}_A$ are the so-called Dirac matrices. These matrices satisfy the following anti-commutation relation

$$(\boldsymbol{e}_a)^A{}_B \, (\boldsymbol{e}_b)^B{}_C \,+\, (\boldsymbol{e}_b)^A{}_B \, (\boldsymbol{e}_a)^B{}_C \,=\, 2 \, g_{ab} \delta^A{}_C \,, \tag{3.111}$$

which is the definition of the Clifford algebra itself. In the same way as (3.107), we can define the action of the connection on this frame of spinors $\{\boldsymbol{\xi}_A\}$ as

$$\hat{\nabla}_a \, \boldsymbol{\xi}_A \equiv (\Omega_a)^B{}_A \boldsymbol{\xi}_B \,, \tag{3.112}$$

where Ω_a is the so-called the spinorial connection. We must find a expression for Ω_a for which the equation (3.108) and (3.110) holds. Then, computing the action of $\hat{\nabla}_a$ on $\boldsymbol{e}_b \, \boldsymbol{\xi}_A$ by means of the equations (3.107) and (3.112) we find that

$$\omega_{ab}{}^c \boldsymbol{e}_c \,=\, \Omega_a \boldsymbol{e}_b \,-\, \boldsymbol{e}_b \Omega_a \,, \tag{3.113}$$

on which the spinorial indices have been omitted for simplicity. One can show that, if we choose the spinorial frame satisfying the equation (3.110), the general solution for this equation is [9]:

$$\Omega_a \,=\, -\frac{1}{4} \omega_a{}^{bc} \, \boldsymbol{e}_b \, \boldsymbol{e}_c \,. \tag{3.114}$$

Thus, if $\boldsymbol{\psi}^A$ are the components of $\boldsymbol{\psi}$ on the frame $\{\boldsymbol{\xi}_A\}$, then an arbitrary spinor field has the following abstract representation

$$\boldsymbol{\psi} \,=\, \boldsymbol{\psi}^A \boldsymbol{\xi}_A \,, \tag{3.115}$$

and its covariant derivative is given by:

$$\hat{\nabla}_a \boldsymbol{\psi} \,=\, \left(\partial_a \psi^A \,+\, (\Omega_a)^A{}_B \psi^B\right) \boldsymbol{\xi}_A \,=\, \left(\partial_a \,-\, \frac{1}{4} \omega_a{}^{bc} \, \boldsymbol{e}_b \, \boldsymbol{e}_c\right) \boldsymbol{\psi} \,. \tag{3.116}$$

In terms of the components $\boldsymbol{\psi}^A$ the latter equation is written as

$$\hat{\nabla}_a \boldsymbol{\psi}^A \,=\, \partial_a \boldsymbol{\psi}^A \,+\, (\Omega_a)^A{}_B \psi^B \,. \tag{3.117}$$

We can also construct a section in a dual spinorial bundle. Let $\{\boldsymbol{\xi}^A\}$ be a frame for the latter bundle. If $\boldsymbol{\chi}_A$ are the components of a dual spinor field $\boldsymbol{\chi}$ on frame $\{\boldsymbol{\xi}^A\}$, then its covariant derivative is given by

$$\hat{\nabla}_a \chi_A \,=\, \partial_a \chi_A \,-\, (\Omega_a)^B{}_A \,\chi_B \,. \tag{3.118}$$

Thus, the operator $\hat{\nabla}_a$ when acting on objects with several spinorial indices follows trivially from the latter equations. Its action on a such object is given by

$$\begin{aligned}
\hat{\nabla}_a \,\Psi^{A_1\ldots A_p}{}_{B_1\ldots B_q} \;&=\; \partial_a \,\Psi^{A_1\ldots A_p}{}_{B_1\ldots B_q} \\
&+\; (\Omega_a)^{A_1}{}_C \,\Psi^{CA_2\ldots A_p}{}_{B_1\ldots B_q} \,+\, (\Omega_a)^{A_2}{}_C \,\Psi^{A_1C\ldots A_p}{}_{B_1\ldots B_q} \,+\, \ldots \\
&-\; \Psi^{A_1\ldots A_p}{}_{CB_2\ldots B_q}(\Omega_a)^C{}_{B_1} \\
&-\; \Psi^{A_1\ldots A_p}{}_{B_1C\ldots B_q}(\Omega_a)^C{}_{B_2} \,-\, \ldots\,.
\end{aligned} \tag{3.119}$$

In particular, using the equations (3.112) and (3.113) one can easily prove that the action of operator $\hat{\nabla}_a$ on a vector field $\boldsymbol{V} = V^b \boldsymbol{e}_b$ lead us to the following result:

$$\hat{\nabla}_a \boldsymbol{V} \,=\, (\partial_a V^b)\boldsymbol{e}_b \,+\, V^b(\Omega_a \,\boldsymbol{e}_b \,-\, \boldsymbol{e}_b \,\Omega_a) \,=\, \partial_a \boldsymbol{V} \,+\, [\Omega_a, \boldsymbol{V}]\,. \tag{3.120}$$

where $[\Omega_a, \boldsymbol{V}] = \Omega_a \boldsymbol{V} - \boldsymbol{V}\Omega_a$ is the commutator of the Clifford algebra. It means that, in spinorial language, a vector is an object with two spinorial indices. More precisely, each vectorial index is equivalent to two spinorial indices, one up and one down. In particular, in general relativity, the spinorial formalism of 4-dimensional Lorentzian manifolds furnishes two types of indices, the ones associated with Weyl spinors of positive chirality and ones related to Weyl spinors of negative chirality. In this case, a vectorial index is equivalent to the product of two spinorial indices, one of positive chirality and one of negative chirality. This is the principle of the Penrose's method [32, 31, 23].

Now, since the Riemann tensor measures the curvature of a tangent bundle, we can relate the curvature of the spinorial connection with the Riemann tensor. First, note that the vector fields $\boldsymbol{e}_a$ are written in terms of the coordinate frames $\boldsymbol{e}_\mu = \partial_\mu$. In this frame, the spinorial representation of the coordinate frame $\boldsymbol{e}_\mu$ will be denoted by the matrices Γ_μ,

$$\boldsymbol{e}_\mu \,\boldsymbol{\xi}_A \equiv (\Gamma_\mu)^B{}_A \boldsymbol{\xi}_B \,. \tag{3.121}$$

Although, sometimes, it is convenient to omit the spinorial indices, when we compute the covariant derivative of Γ_μ, we must account for the covariant derivative of both the vectorial index μ as well as the omitted spinorial indices. Taking this into account and noting that these contributions cancel each other we eventually obtain: $\hat{\nabla}_\mu \Gamma_\nu = 0$. The curvature of the spinorial bundle is defined by the action of the commutator $[\hat{\nabla}_\mu, \hat{\nabla}_\nu]$ on spinor $\boldsymbol{\psi}$ and its relation with the Riemann tensor is given by [1]:

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]\,\boldsymbol{\psi} \,=\, \frac{1}{4} R_{\mu\nu}{}^{\rho\sigma}\Gamma_{\rho\sigma}\boldsymbol{\psi} \,, \tag{3.122}$$

where $\Gamma_{\mu\nu} \equiv \frac{1}{2}[\Gamma_\mu, \Gamma_\nu]$. The spinorial space is defined to be the space where an irreducible and minimal representation of the Clifford algebra acts. Then, if $\boldsymbol{e}_\mu$ is a section whose spinorial representation is the matrix Γ_μ, we can define an operator D as $D = \Gamma^\mu \hat{\nabla}_\mu$, called Dirac operator. A spinor field $\boldsymbol{\psi}$ satisfy the following equation:

$$D\,\boldsymbol{\psi} = m\boldsymbol{\psi}\,, \tag{3.123}$$

where m is some complex constant. What, in fact, determine if the eigenvalue m is real or imaginary is the nature of the manifold. After some algebra, we can prove that

$$D^2\boldsymbol{\psi} \;=\; \left(\hat{\nabla}^\mu\hat{\nabla}_\mu - \frac{1}{4}R\right)\boldsymbol{\psi}\,, \tag{3.124}$$

where R is the curvature scalar. The square of the Dirac operator is known as the spinorial Laplacian. Finally, if we define the constant α by the following relation:

$$\hat{\nabla}_\mu\boldsymbol{\psi} \;=\; \alpha\Gamma_\mu\boldsymbol{\psi} \tag{3.125}$$

the corresponding spinor field $\boldsymbol{\psi}$ is a Killing spinor field. The constant α is known as Killing constant. Note that by means of (3.123), a Killing spinor field is a Dirac spinor field with $m = \alpha n$, but the converse is not true. In practical terms, the utility of the Killing spinor fields is that they can be used to generate symmetry tensors which eventually lead to conservation laws as well as to the integrability of field equations using the manifold as a background [9]. Indeed, if Υ is the chiral matrix, $\boldsymbol{\psi}$ is a Killing spinor and $\overline{\boldsymbol{\psi}}$ its conjugate then each of the tensors [35, 36, 37]

$$\begin{array}{lcl} <\boldsymbol{\psi}, \Gamma_{\mu_1\mu_2\ldots\mu_p}\boldsymbol{\psi}> & , & <\boldsymbol{\psi}, \Gamma_{\mu_1\mu_2\ldots\mu_p}\,\Upsilon\,\boldsymbol{\psi}> \\ <\overline{\boldsymbol{\psi}}, \Gamma_{\mu_1\mu_2\ldots\mu_p}\boldsymbol{\psi}> & , & <\overline{\boldsymbol{\psi}}, \Gamma_{\mu_1\mu_2\ldots\mu_p}\,\Upsilon\,\boldsymbol{\psi}>\,, \end{array} \tag{3.126}$$

is either a Killing-Yano (KY) tensor or a closed conformal Killing-Yano (CCKY) tensor [37, 38].

4. Monogenics and Spinors in Curved Spaces

In the previous chapters the Clifford algebra was introduced in the geometry, that is the reason why the Clifford algebra is also called the geometric algebra. Now, it is well known that holomorphic functions, also called conformal maps, are the central objects of study in complex analysis [39]. Holomorphic functions of complex variables satisfy Cauchy-Riemann's equations and, hence, they also satisfy Laplace's equation [2]. In this chapter we introduce some basic concepts in Clifford analysis, the Clifford algebra in analysis, which furnish one approach of the complex analysis in dimensions higher than two. One of the main subjects studied in Clifford analysis is the function theory of monogenic functions and its interaction with the representation theory of the group $Spin(m)$. In Clifford analysis one consider multivector functions that solve Cauchy-Riemann or Dirac equations on some manifolds. In particular, the idea of higher dimensional holomorphic functions is given by the so-called **monogenic functions** which are multivector functions that are annihilated from the left by the Dirac operator. Also, we will show that the Dirac equation minimally coupled to an electromagnetic field is separable in spaces that are the direct product of bidimensional spaces. In particular, we applied on the background of black holes whose horizons have topology $\mathbb{R} \times S^2 \times \ldots \times S^2$.

4.1 Monogenics in the Literature

For a review of many of the basic properties of Clifford analysis including some historical remarks we refer to [42, 43]. Clifford analysis may also be regarded as a special function theory within harmonic analysis since the Dirac operator factorizes the Laplace operator, so it may be seen as a refinement of harmonic analysis. One important tool in the analysis of the Dirac operator is the Cauchy integral formula. This formula allows us to calculate the values of functions in terms of given boundary data arising from practical measurements. Based on these representations important existence and uniqueness theorems for the solutions of boundary value problems on manifolds could be established. The so-called monogenic functions are those that are in the **kernel** of the Dirac operator and they can be seen as higher dimensional holomorphic functions, they also satisfy the Laplace's equation. In the simplest case of one dimensional space a monogenic function is a function with null derivative, that is, a constant with no structure; in two dimensional space monogenic functions are equivalent to analytic functions of a complex variable and increasing the space dimensionality one finds that those functions can generate many kinds of interesting geometrical structures. The monogenic functions were introduced by F. Brackx, R. Delanghe and F. Sommen in [46] and in the past forty years have been successfully and intensively studied. The literature is very rich of results but studies on the topic are ongoing, see [47, 48, 49]. In [50] the monogenic functions were applied on some conformally flat manifolds such as cylinders, tori and some conformally flat manifolds of genus bigger or equal to two. Lounesto obtains in reference [2] the monogenic functions on flat space in arbitrary dimension and emphasizes that the monogenic functions take their value in a Clifford algebra which is a natural environment to represent internal degrees of freedom of elementary particles such as spin.

It is well known that the spherical harmonics play an important role in the harmonic analysis of the Laplace operator, in Clifford analysis a similar role is played by spherical monogenics, also called spin weighted spherical harmonics, which are polynomial solutions of the Dirac equation without mass defined on the sphere. These functions form a complete orthonormal set for each value of spin weight and are a refinement of the notion of spherical harmonics. In reference [51] was made a study of the generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in arbitrary dimensions. The spin weighted spherical harmonics were introduced by Newman and Penrose in [53] as a means to describe certain quantities exhibiting a particular "spin-gauge" behavior. For instance, for particular choice of spin-gauge, the spin-weighted spherical harmonics reduce to the monopole harmonics which arise as solutions of the Schrodinger equation for an electron in the field of a magnetic monopole introduced by Wu and Yangin in [54], see also [23, 55, 56]. Several angular functions, including scalar, vector, and tensor spherical harmonics, are used to perform separation of variables in the general relativity literature. These functions include the Regge-Wheeler harmonics, the symmetric, trace-free tensors of Sachs and Pirani, the Newman-Penrose spin weighted spherical harmonics. A good review article of these functions by Thorne is the reference [57]. The spin weighted spherical harmonics can be expressed in terms of the Wigner matrix since its formulas in spherical coordinates are identical to formulas for Wigner matrices for certain values of Euler angles, also in terms of the Jacobi Polynomials, the generalized associated Legendre functions and the hypergeometric functions [58]. Applications of these functions in the solution by separation of variables of various systems of partial differential equations includes source-free Maxwell equations in flat space-time and in the Schwarzschild spacetime, the Einstein vacuum field equations linearized about the flat space-time and the Dirac equation [59]. Finally, to solve the Teukolsky Master Equation, which describes the dynamics of various fields of different spins as perturbations to Kerr metric, by separation of variables there are two non-trivial equations obtained that are the angular equation and the radial equation whose solutions for the angular equation are the called spin weighted spheroidal harmonics, see [60, 61, 62, 63, 64] for more details, introduced by Teukolsky in the context of black hole physics, which can be reduced to spin weighted spherical harmonics for a particular case. So, the spin weighted spherical harmonics have innumerable applications in several branches of the physics as was mentioned.

4.2 Monogenic Multivector Functions : Definitions and Operator Equalities

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ be an orthonormal frame of Euclidean space $\mathbb{R}^{p,q}$ of dimension $n = p+q$ endowed with a non-degenerate symmetric metric of signature $s = |p-q|$ and $\mathcal{C}\ell_{p,q}$ the Clifford algebra over $\mathbb{R}^{p,q}$. It is possible to extract a certain kind of square root of the n-dimensional Laplacian operator Δ and consider instead a first-order differential operator, called the Dirac operator $\boldsymbol{D}$. The Dirac operator gets its name from its appearance in Dirac's wave equation for the electron. With respect to this basis, the Dirac operator on $\mathbb{R}^{n,0}$ is given by

$$\boldsymbol{D} = \boldsymbol{e}_1\partial_1 + \boldsymbol{e}_2\partial_2 + \ldots + \boldsymbol{e}_n\partial_n = \delta^{ab}\boldsymbol{e}_a\partial_b \quad , \quad \text{a,b} = 1, 2, \ldots, n\,, \tag{4.1}$$

where ∂_a represents the partial derivative with respect to the variable x^a. If we want the Dirac operator to be the square root of Laplacian, $\boldsymbol{DD} = \Delta$, the elements of the frame $\{\boldsymbol{e}_a\}$ must obey the following algebra

$$\boldsymbol{e}_a\boldsymbol{e}_b + \boldsymbol{e}_b\boldsymbol{e}_a = \boldsymbol{g}(\boldsymbol{e}_a, \boldsymbol{e}_b)\,, \tag{4.2}$$

which is just the very definition of Clifford algebra $\mathcal{C}\ell_{n,0}$. It means that the Dirac operator is seen as a vector in $\mathcal{C}\ell_{n,0}$. An important variation of the Dirac operator which provides a closer contact with the classical function theory, denoted by $\boldsymbol{D}_W$, is the Weyl (or Cauchy-Riemann) operator obtained as follows. Let $\boldsymbol{X}$ be a vector of the Euclidean space $\mathbb{R}^{n,0}$, $\boldsymbol{X} = X^a\boldsymbol{e}_a$. We can take a special vector of the frame $\{\boldsymbol{e}_a\}$, $\boldsymbol{e}_1$ for example, and then consider its left Clifford multiplication on the vector $\boldsymbol{X}$, that is,

$$\begin{aligned} \boldsymbol{Z} &= \boldsymbol{e}_1\boldsymbol{X} = X^a\boldsymbol{e}_1\boldsymbol{e}_a \\ &= X_1 + X_2\boldsymbol{e}_1\boldsymbol{e}_2 + \ldots + X_n\boldsymbol{e}_1\boldsymbol{e}_n\,, \end{aligned} \tag{4.3}$$

where the elements $\boldsymbol{e}_1\boldsymbol{e}_a\,(a = 2, 3, \ldots, n)$ are unit bivetors. The combination of a scalar and a bivector, which is formed naturally via the Clifford product, is therefore a multivector of the $\mathcal{C}\ell_{n,0}^{+}$, the even subalgebra of $\mathcal{C}\ell_{n,0}$, which is generated by the set $\{1, \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_1\boldsymbol{e}_3, \ldots, \boldsymbol{e}_1\boldsymbol{e}_n\}$. In two dimensions, for example, defining $X_1 = x, X_2 = y$, the element $\boldsymbol{Z} = x + \boldsymbol{I}y$ can be viewed as a complex number, since the bivector $\boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_2$, the pseudoscalar of $\mathcal{C}\ell_{2,0}^{+}$, is such that $\boldsymbol{I}^2 = -1$ and can be viewed as the unit imaginary i. The complex number $Z = x + iv$ represents a vector in the complex plane, with Cartesian components x and y. Since the bivectors $\boldsymbol{e}_1\boldsymbol{e}_a$ anticommute and square equal to -1, they form an orthonormal basis for the vector space $\mathbb{R}^{0,n-1}$. Thus, the even subalgebra $\mathcal{C}\ell_{n,0}^{+}$ is isomorphic to $\mathcal{C}\ell_{0,n-1}$. Defining $\boldsymbol{i}_a = \boldsymbol{e}_1\boldsymbol{e}_a$, a vector $\boldsymbol{X} \in \mathbb{R}^{n,0}$ can be replaced by a sum of a vector and a scalar. Such an element will be called a paravector $\boldsymbol{Z} \in \mathbb{R} \oplus \mathbb{R}^{0,n-1}$,

$$\boldsymbol{Z} = X_1 + X_2\boldsymbol{i}_2 + \ldots + X_n\boldsymbol{i}_n = X_1 + \boldsymbol{X}_1 , \tag{4.4}$$

where $\boldsymbol{X}_1 = X_2\boldsymbol{i}_2 + \ldots + X_n\boldsymbol{i}_n$ is a vector on the vector space $\mathbb{R}^{0,n-1}$. Therefore, there is a natural map between $\boldsymbol{X}$ and $\boldsymbol{Z}$ achieved by means of multiplying the first by $\boldsymbol{e}_1$ the left. The role of the preferred vector $\boldsymbol{e}_1$ is clear, it is the "real axis" of the higher dimensional analogue of complex analysis. From the geometrical point of view, we could say that when using the paravector formalism, we have chosen a "real axis" namely the $\boldsymbol{e}_1$ axis by multiplying the vector $\boldsymbol{X} \in \mathcal{C}\ell_{n,0}$ on the left by $\boldsymbol{e}_1$. This established a correspondence between the following two mappings [2]:

$$\begin{array}{ccc} f : \mathbb{R}^{n,0} & \to & \mathcal{C}\ell_{n,0}^{+} \\ \boldsymbol{X} & \mapsto & f(\boldsymbol{X}) \end{array} , \qquad \begin{array}{ccc} f : \mathbb{R} \oplus \mathbb{R}^{0,n-1} & \to & \mathcal{C}\ell_{0,n-1} \\ \boldsymbol{Z} & \mapsto & f(\boldsymbol{Z}) \end{array} . \tag{4.5}$$

For convenience, we denote both by f. In this way, the Dirac operator is replaced by the Weyl operator $\boldsymbol{D}_W$ on $\mathbb{R} \oplus \mathbb{R}^{0,n-1}$ defined by

$$\boldsymbol{D}_W = \partial_1 + \boldsymbol{i}_2\partial_2 + \boldsymbol{i}_3\partial_3 + \ldots + \boldsymbol{i}_n\partial_n = \partial_1 + \boldsymbol{D}_1 , \tag{4.6}$$

where here $\boldsymbol{D}_1$ is the Dirac operator on $\mathbb{R}^{0,n-1}$. A function $f : U \subset \mathbb{R}^{p,q} \to \mathcal{C}\ell_{p,q}$ is seen as a function $f(\boldsymbol{X})$ of $\boldsymbol{X} \in \mathbb{R}^{p,q}$. There are in the literature several ways to define a notion of generalized holomorphic function with values in the Clifford algebra. The most successful is the so-called monogenicity which has been intensively studied during the past forty years. Formally, let $\mathcal{V}$ be a vector space over a real or complex field $\mathbb{F}$. A smooth multivector function $f(\boldsymbol{X})$ defined on an open set $U \subset \mathcal{V}$ and having values in the Clifford algebra $\mathcal{C}\ell(\mathcal{V})$ is said to be monogenic if and only if it is in the kernel of the Dirac operator on $\mathcal{V}$

$$\boldsymbol{D}f(\boldsymbol{X}) = 0 , \tag{4.7}$$

for each $\boldsymbol{X} \in U$. The theory of the functions in the kernel of the Dirac operator or of the Weyl operator are equivalent and the word monogenic is used for functions in the kernel of either of them. One of the simplest example is the case $n = 2$ on which the Clifford algebra $\mathcal{C}\ell_{0,1}$ is isomorphic to the complex field $\mathbb{C}$. In this case, the Weyl operator is given explicitly by $\boldsymbol{D}_W = \frac{\partial}{\partial x} + \boldsymbol{i}\frac{\partial}{\partial y}$ and a multivector function $f(\boldsymbol{Z}) = u(x,y) + \boldsymbol{i}\, v(x,y) \in \mathcal{C}\ell_{0,1}$ of $\boldsymbol{Z} = x + \boldsymbol{i}y$, which is equivalent to analytic functions, is monogenic if and only if the following equations of holomorphy hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\,. \tag{4.8}$$

These equations are the well-known Cauchy-Riemann equations. The statement that $f(\boldsymbol{Z})$ is an analytic function, i.e., a function that satisfies the Cauchy-Riemann equations, reduces to a monogenic equation. In the case $n = 3$, the Clifford algebra $\mathcal{C}\ell_{0,2}$ corresponds to the quaternion algebra $\mathbb{H}$. The Weyl operator is given by $\boldsymbol{D}_W = \frac{\partial}{\partial x} + \boldsymbol{i}\frac{\partial}{\partial y} + \boldsymbol{j}\frac{\partial}{\partial z} + \boldsymbol{k}\frac{\partial}{\partial w}$. Then, identifying $\boldsymbol{i} = \boldsymbol{i}_1, \boldsymbol{j} = \boldsymbol{i}_2, \boldsymbol{k} = \boldsymbol{i}_3$, the quaternion $\boldsymbol{Q} = q_x + q_y\boldsymbol{i} + q_z\boldsymbol{j} + q_w\boldsymbol{k} \in \mathbb{H}$ is monogenic in U if $\boldsymbol{D}_W\boldsymbol{Q} = 0$, thus giving rise to a **generalized Cauchy-Riemann system**. Note that the holomorphic functions are easily transferable to higher dimensions when considering Clifford algebra.

In the next sections, we will offer an approach of monogenic sections on manifolds with values in the Clifford bundle or Spinor bundle. A lot of the formulas thus established for the Dirac operator remain valid in the case of the Weyl operator $\boldsymbol{D}_W$ by formally replacing $\boldsymbol{e}_1$ by 1 and $\boldsymbol{e}_a$ by $\boldsymbol{i}_a\,, a = 2, \ldots, n$.

4.3 Monogenics in Curved Spaces and Some Results

In the section 2.2 we identified the Clifford algebra with the vector space of multivectors $\wedge\mathcal{V}$ with the Clifford product between vectors given in (2.14), namely the following algebra holds

$$\boldsymbol{U}\boldsymbol{V} + \boldsymbol{U}\boldsymbol{V} = 2\,\boldsymbol{g}(\boldsymbol{V},\boldsymbol{U}) \quad \forall\, \boldsymbol{V},\boldsymbol{U} \in \mathcal{V}\,.$$

In the context of bundles, the vectors are elements of $\Gamma(\mathcal{V})$, the space of sections of the bundle $\mathcal{V}$. So, in this approach a multivector is a multisection of the exterior bundle $\wedge\mathcal{V}$. Since we are concerned only with local results, we are allowed to identify the complexified tangent spaces $\mathbb{C} \otimes T_qM$, at point $q \in M$, with a vector space $\mathcal{V}$, so that all the results of the previous chapters can be used. Thus, it follows that on a manifold $(M, \boldsymbol{g})$ we have the structure of a Clifford algebra on each fibre of $\wedge M = \bigcup_p \wedge_p M$, the exterior bundle. The exterior bundle $\wedge M$ equipped with the Clifford product in the fibres will be called the Clifford bundle and denoted by $\mathcal{C}M$. We use the term Clifford forms refer to the multisections as elements of the Clifford bundle rather than exterior algebra. The set of the Clifford forms on $\mathcal{C}M$ is denoted by $\Gamma(\mathcal{C}M)$.

In a formal way, in curved spaces, the Dirac operator $\boldsymbol{D} = \boldsymbol{e}^a\nabla_a$ is a first order differential operator that acts on a vector bundle over the manifold $(M, \boldsymbol{g})$. The study of Dirac and Laplace operators on manifolds, in particular on Riemannian manifolds, has lead to a profound understanding of many geometric aspects related to these manifolds, in particular some applications for the Dirac operators on manifolds have been developed in [41, 44, 45, 40]. In turn, Riemannian manifolds play a central role in several branches of modern physics. They appear in important cosmological models, in general relativity theory, in the standard model of particle physics, in string theory and in general quantum field theory [23, 19, 33]. Here, we consider only Clifford forms $\mathcal{A}$ whose components are smooth, that is, its partial derivatives up to a certain order are all continuous in some domain U of $(M, \boldsymbol{g})$. Occasionally we will focus on smooth sections $\boldsymbol{V} = V^a \boldsymbol{e}_a \in \Gamma(TM)$ of the bundle TM on which ∇_a is the Levi-Civita connection. Saying that a section $\boldsymbol{V} \in \Gamma(TM)$ is monogenic, which means that Clifford action of the Dirac operator $\boldsymbol{D}$ on $\boldsymbol{V}$ is null,

$$\boldsymbol{D}\boldsymbol{V} = 0\,,$$

is equivalent to saying that its components $V^a\,, (a = 1, ..., n)$ satisfy the following system of equations

$$\begin{aligned} \boldsymbol{D}\boldsymbol{V} &= <\boldsymbol{D}, \boldsymbol{V}> + \boldsymbol{D}\wedge\boldsymbol{V} = \nabla_a V^a + \nabla^{[a}V^{b]}\boldsymbol{e}_a \wedge \boldsymbol{e}_b = 0 \\ &\Rightarrow \nabla_a V^a = 0 \quad \text{and} \quad \nabla^{[a}V^{b]} = 0\,. \end{aligned} \tag{4.9}$$

When restricted to the flat space, the equations (4.9) are the generalized Cauchy-Riemann equations $\partial_a V^a = 0$ and $\partial^{[a}V^{b]} = 0$. Introducing the co-derivative operator, $\delta : \Gamma(\wedge_p M) \to \Gamma(\wedge_{p-1} M)$, defined by the action on Clifford forms as:

$$\delta\boldsymbol{\omega} = \star^{-1} d \star \widehat{\boldsymbol{\omega}} \quad , \quad \boldsymbol{\omega} \in \Gamma(\wedge_p M)\,, \tag{4.10}$$

with $\star$ being the dual Hodge, d the exterior derivative and $\widehat{\boldsymbol{\omega}}$ is the degree involution of a p-vector $\boldsymbol{\omega} \in \Gamma(\wedge_p M)$, the equation (4.9) can be conveniently written as follows:

$$\delta\boldsymbol{V} = 0 \quad \text{and} \quad d\boldsymbol{V} = 0\,. \tag{4.11}$$

By the Poincarè lemma, locally we can assert that $d\boldsymbol{V} = 0$ implies that in some neighborhood of every point p on M there exists a function ϕ such that $\boldsymbol{V} = d\phi$, hence $\delta\boldsymbol{V} = \delta d\phi = 0$. Note that, while d increases the degree of a Clifford form by one, δ decreases the degree by one. This implies that $d\delta\phi = 0$, since $\delta\phi = 0$ by definition. So, the action of the operator $(d\delta + \delta d)$ on $\phi = 0$ also vanishes. This operator is exactly the negative of the Laplacian operator

$$\Delta = -(d\delta + \delta d)\,. \tag{4.12}$$

Thus, the function ϕ is said to be harmonic, that is satisfy $\Delta\phi = 0$. Therefore, given $\phi \in \Gamma(\wedge_0 M)$ such that $\Delta\phi = 0$, any monogenic section $\boldsymbol{V}$ is obtained by judiciously choosing

$$\boldsymbol{V} = d\phi \quad \forall \quad \boldsymbol{V} \in \Gamma(TM)\,. \tag{4.13}$$

The previous results about monogenicity can be generalized immediately to higher dimensions. These generalisations invariably turn out to be of mathematical and physical importance, and it is no exaggeration to say that equations of the type of equation (4.7) are amongst the most studied in physics.

Instead of sections $\boldsymbol{V} \in \Gamma(TM)$, we can be more ambitious and pass to consider more general elements $\mathcal{A} \in \Gamma(\wedge M)$ of the Clifford bundle $\wedge M$. Let us consider a Clifford form of degree p, $\boldsymbol{F} = F^{a_1 a_2 \ldots a_p} \boldsymbol{e}_{a_1} \wedge \boldsymbol{e}_{a_2} \wedge \ldots \wedge \boldsymbol{e}_{a_p} \in \Gamma(\wedge_p M)$, with $F^{a_1 a_2 \ldots a_p} = F^{[a_1 a_2 \ldots a_p]}$ totally skew-symmetric a section on $\Gamma(\wedge_p M)$. Then, the element $\boldsymbol{F}$ is monogenic if the following equation holds:

$$\begin{aligned} \boldsymbol{D F} &= < \boldsymbol{D}, \boldsymbol{F} > + \boldsymbol{D} \wedge \boldsymbol{F} \\ &= \nabla_a F^{a_1 a_2 \ldots a_p} < \boldsymbol{e}_a, \boldsymbol{e}_{a_1} \wedge \boldsymbol{e}_{a_2} \wedge \ldots \wedge \boldsymbol{e}_{a_p} > \\ &+ \nabla^{[a} F^{a_1 a_2 \ldots a_p]} \boldsymbol{e}_{a_1} \wedge \boldsymbol{e}_{a_2} \wedge \ldots \wedge \boldsymbol{e}_{a_p} = 0 \,. \end{aligned} \tag{4.14}$$

Using an important relation obtained in the equation (2.23) which we express here as

$$\begin{aligned} & < \boldsymbol{e}_a, \boldsymbol{e}_{a_1} \wedge \boldsymbol{e}_{a_2} \wedge \ldots \wedge \boldsymbol{e}_{a_p} >= \\ & \sum_{i=1}^{p} (-1)^{i+1} < \boldsymbol{e}_a, \boldsymbol{e}_{a_i} > \boldsymbol{e}_{a_1} \wedge \ldots \wedge \check{\boldsymbol{e}}_{a_i} \wedge \ldots \wedge \boldsymbol{e}_{a_p} \,, \end{aligned} \tag{4.15}$$

where the check on $\check{\boldsymbol{e}}_{a_i}$ select the term will be removed from the series when the sum is expanded, we can easily prove that $\boldsymbol{F} \in \Gamma(\wedge_p M)$ is said to be a monogenic section if and only if

$$\begin{array}{lcl} \nabla_a F^{a a_2 \ldots, a_p} = 0 & \Leftrightarrow & \delta \boldsymbol{F} = 0 \\ \nabla^{[a} F^{a_1 a_2 \ldots, a_p]} = 0 & \Leftrightarrow & d\boldsymbol{F} = 0 \end{array} \,. \tag{4.16}$$

Many equations in physics can be elegantly formulated in terms of the exterior derivative d and the co-derivative δ. In particular, letting $p = 2$ the equations $\delta \boldsymbol{F} = 0$ and $d\boldsymbol{F} = 0$ are the source-free Maxwell's equations which rise from the itself definition of monogenic sections. In this case, the section $\boldsymbol{F} = F^{ab} \boldsymbol{e}_a \wedge \boldsymbol{e}_b \in \Gamma(\wedge_2 M)$, which represents the electromagnetic field, is nothing more than the exterior derivative of the section $\boldsymbol{A} = A^a \boldsymbol{e}_a \in \Gamma(TM)$, $\boldsymbol{F} = d\boldsymbol{A}$.

Instead of sections $\boldsymbol{F} \in \Gamma(\wedge_p M)$, we could instead consider the sections on the subset $\Gamma(\mathcal{C}^+ M)$ of $\Gamma(\mathcal{C} M)$, the space of the even sections of the even sub-bundle $\mathcal{C}^+ M$. The bundle $\mathcal{C}^+ M$ is a direct sum of $\wedge_p M$ with even p. If we require an even section $\mathcal{F} = \sum_p \langle \mathcal{F} \rangle_p = \sum_p \mathcal{F}_p \in \Gamma(\mathcal{C}^+ M)$, where $\mathcal{F}_p = F^{a_1 a_2 \ldots a_p} \boldsymbol{e}_{a_1} \wedge \boldsymbol{e}_{a_2} \wedge \ldots \wedge \boldsymbol{e}_{a_p}$, to be monogenic,

$$\begin{aligned} \boldsymbol{D} \mathcal{F} &= \sum_p \boldsymbol{D} \mathcal{F}_p \\ &= \sum_p (< \boldsymbol{D}, \mathcal{F}_p > + \boldsymbol{D} \wedge \mathcal{F}_p) \,=\, 0 \,, \end{aligned} \tag{4.17}$$

then we have a system of coupled equations:

$$< \boldsymbol{D}, \mathcal{F}_p > + \boldsymbol{D} \wedge \mathcal{F}_{p-2} = 0 \quad , \quad < \boldsymbol{D}, \mathcal{F}_{p+2} > + \boldsymbol{D} \wedge \mathcal{F}_p = 0 \,, \tag{4.18}$$

where $\mathcal{F}_p$ is the homogeneous part of degree p of $\mathcal{F}$. Note that, if a Clifford form $\mathcal{A}$ contains all grades it is clear that both the homogeneous part of even degree and odd degree must be monogenics independently. Without loss of generality, we can therefore assume that $\mathcal{A}$ has even degree. Neverless, if we consider an arbitrary Clifford form $\mathcal{A} \in \Gamma(\mathcal{C}M)$ and its decomposition in pure degree terms

$$\begin{aligned} \mathcal{A} &= A + A^a \boldsymbol{e}_a + A^{ab} \boldsymbol{e}_a \boldsymbol{e}_b + A^{abc} \boldsymbol{e}_a \boldsymbol{e}_b \boldsymbol{e}_c + \ldots + A^{a_1 \ldots a_n} \boldsymbol{e}_{a_1} \ldots \boldsymbol{e}_{a_n} \\ &= \sum_{p=0}^{n} \langle \mathcal{A} \rangle_p = \sum_{p=0}^{n} \mathcal{A}_p \end{aligned} \tag{4.19}$$

Then claiming that $\boldsymbol{D}\mathcal{A}_p = 0$, we obtain

$$\sum_{p=0}^{n} (< \boldsymbol{D}, \mathcal{A}_p > + \boldsymbol{D} \wedge \mathcal{A}_p) \,=\, 0 \,. \tag{4.20}$$

Hence, $\mathcal{A}$ satisfies the condition of monogenicity, $\boldsymbol{D}\mathcal{A} = 0$, if and only if

$$< \boldsymbol{D}, \mathcal{A}_{p+1} > + \boldsymbol{D} \wedge \mathcal{A}_{p-1} \,=\, 0 \quad , \quad p \,=\, 0, 1, \ldots, n \,, \tag{4.21}$$

with $\mathcal{A}_{-1} = \mathcal{A}_{n+1} = 0$. The main difference with complex analysis is that we cannot derive new monogenics simply from power series in this solution, due to non-commutativity. Instead, we can construct monogenic functions from series of geometric products, but a more instructive route is to classify monogenics via their angular properties, but in this dissertation we do not analyze this approach. See [4, 2, 40, 51] for a more detailed study.

Instead of Clifford forms on the Clifford bundle, in the next section, we will examine the solutions lying in minimal left ideals, these carrying irreducible representations of the Clifford algebra.

4.4 Monogenic Equation for Spinor Fields

In section 2.4 spinors were defined as the elements that generate a vector space that provides a faithful representation for the spin group and carry an irreducible representation of the Clifford algebra. Any such irreducible representation is equivalent to that carried by a minimal left ideal of the Clifford algebra, the called spinorial representation. Hence we thus took the minimal left ideals as the space of spinors, see subsection 2.4.1. The Clifford bundle of a manifold $(M, \boldsymbol{g})$ has as fibre at p, the Clifford algebra of the tangent space of $(M, \boldsymbol{g})$ at p the Clifford algebra $\mathcal{C}\ell(T_pM)$.

Any minimal left ideal of this fibre algebra carries the spinor representation. A manifold $(M, \boldsymbol{g})$ is said to be a spin manifold if it has a well-defined spin structure. However, a spin manifold can allow different spin structures. Then, in the following, when talking about a d-dimensional spin manifold $(M, \boldsymbol{g})$ it will always be assumed that a spin structure is already chosen and fixed. This means that we can smoothly assign a minimal left ideal of the fibre algebra to each p in $(M, \boldsymbol{g})$, then we have a bundle over $(M, \boldsymbol{g})$ with each fibre carrying an irreducible representation of the corresponding fibre of the Clifford bundle. This is the spinor bundle SM, sections being the spinor fields on $\Gamma(SM)$, the space of the sections of SM. Such a bundle of spinor spaces is a sub-bundle of the Clifford bundle. Note that the spin structure, the spinor bundle and the Dirac operator depend on the metric $\boldsymbol{g}$ of M.

Below, let us investigate the behavior of the Dirac operator under **conformal transformation**. In order to accomplish this, let us introduce a local coordinate system $\{x^\mu\}\,(\mu = 0, 1, \ldots, d)$ on the manifold $(M, \boldsymbol{g})$. The line element is written as $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$, where $g_{\mu\nu} = \boldsymbol{g}(\partial_\mu, \partial_\nu)$ are the components of $\boldsymbol{g}$ in the coordinate frame. However, given an orthonormal tangent frame $\{\boldsymbol{e}_\alpha\}$, we can define the dual frame of 1-forms $\{\boldsymbol{e}^\alpha\}$ defined by $\boldsymbol{e}^\alpha(\boldsymbol{e}_\beta) = \delta^\alpha{}_\beta$ and such that the previous line element can be written as

$$ds^2 = \delta_{\alpha\beta}\, \boldsymbol{e}^\alpha \boldsymbol{e}^\beta \quad \alpha, \beta = 1, \ldots, d\,, \tag{4.22}$$

where $\delta_{\alpha\beta} = \boldsymbol{g}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta)$ are now the constant components of the metric relative to this orthonormal frame. Now, let $\hat{\boldsymbol{g}}$ be a metric that is conformally related to the metric $\boldsymbol{g}$, i.e., there is a function $\Omega \in \mathfrak{F}(M)$ over the manifold with $\Omega > 0$ and $\hat{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}$. Then, if $\{\hat{\boldsymbol{e}}_\alpha\}$ is an orthonormal frame with respect to the metric $\hat{\boldsymbol{g}}$ the following relations hold

$$\hat{\boldsymbol{e}}^\alpha = \Omega\, \boldsymbol{e}^\alpha \quad , \quad \hat{\boldsymbol{e}}_\alpha = \frac{1}{\Omega}\, \boldsymbol{e}_\alpha\,. \tag{4.23}$$

Remember that, the derivatives of the frame vector fields $\boldsymbol{e}_\alpha$ determine the coefficients of the spin connection according to the equation (3.44), namely $\nabla_\alpha \boldsymbol{e}_\beta = \omega_{\alpha\beta}{}^{\varepsilon} \boldsymbol{e}_\varepsilon$. Since we are dealing with metric-compatible connections, a change of metric leads to a different spin connection. With respect to the metric $\hat{\boldsymbol{g}}$ it is given by the following relation

$$\hat{\nabla}_\alpha \hat{\boldsymbol{e}}_\beta = \hat{\omega}_{\alpha\beta}{}^{\varepsilon} \hat{\boldsymbol{e}}_\varepsilon\,. \tag{4.24}$$

Given the frames of 1-forms $\{\hat{\boldsymbol{e}}^\alpha\}$, we recall that the connection 1-forms $\hat{\boldsymbol{\omega}}^\alpha{}_\beta$ are determined by the first Cartan structure equation and since the relation between the frames $\{\hat{\boldsymbol{e}}^\alpha\}$ and $\{\boldsymbol{e}^\alpha\}$ is known, we can find the relation between the two spin connection one-forms $\hat{\boldsymbol{\omega}}^\alpha{}_\beta$ corresponding to the frame of 1-forms $\hat{\boldsymbol{e}}^\alpha$ and $\boldsymbol{\omega}^\alpha{}_\beta$ relative to the frame $\boldsymbol{e}^\alpha$. Indeed, after some algebra, from the equation

$$d\hat{\boldsymbol{e}}^\alpha + \hat{\boldsymbol{\omega}}^\alpha{}_\beta \wedge \hat{\boldsymbol{e}}^\beta = 0\,,$$

using (4.23) we eventually arrive at the following expression:

$$\hat{\boldsymbol{\omega}}^{\alpha}{}_{\beta} = \boldsymbol{\omega}^{\alpha}{}_{\beta} + \frac{1}{\Omega}\,(\,\partial_{\beta}\Omega\,\boldsymbol{e}^{\alpha} - \partial_{\alpha}\Omega\,\boldsymbol{e}^{\beta}\,)\,. \tag{4.25}$$

Defining $\boldsymbol{\omega}^{\varepsilon}{}_{\beta}(\boldsymbol{e}_{\alpha}) \equiv \omega_{\alpha\beta}{}^{\varepsilon}$ and using the expression $\boldsymbol{e}^{\alpha}(\boldsymbol{e}_{\beta}) = \delta^{\alpha}{}_{\beta}$, from the equation (4.25) we see that the coefficients of the spin connection 1-forms $\hat{\omega}_{\alpha\beta}{}^{\varepsilon}$ and $\omega_{\alpha\beta}{}^{\varepsilon}$ are related by

$$\hat{\omega}_{\alpha}{}^{\beta\varepsilon} = \frac{1}{\Omega}\,\omega_{\alpha}{}^{\beta\varepsilon} + \frac{2}{\Omega^{2}}\,\partial^{[\beta}\Omega\,\delta^{\varepsilon]}{}_{\alpha}\,, \tag{4.26}$$

where indices inside square brackets are anti-symmetrized.

We now consider a conformal transformation of the Dirac operator, which will be of future relevance in the next section. Let $\{\boldsymbol{e}_{\alpha}\}$ be an orthonormal tangent frame for a bundle carrying the irreducible representation of the Clifford bundle, $\boldsymbol{g}(\boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta}) = \delta_{\alpha\beta}$. The **monogenic equation** for a spinor field $\boldsymbol{\psi}$ is

$$\boldsymbol{D}\boldsymbol{\psi} = 0 \quad , \quad \forall\, \boldsymbol{\psi} \in \Gamma(SM)\,, \tag{4.27}$$

where $\boldsymbol{D} = \delta^{\alpha\beta}\boldsymbol{e}_{\alpha}\nabla_{\beta}$ is the Dirac operator and ∇_{α} is the Levi-Civita connection of the spinor bundle, that is the Levi-Civita connection satisfing the Leibniz rule with respect to the Clifford product and with respect to the natural inner products on space of spinorial sections. When we deal with spinors, the physical concepts can be understood in a less abstract way by making use of the Dirac matrices. In even dimension $d = 2n$, the Dirac matrices Γ_{α} represent faithfully the Clifford algebra by $2^{n} \times 2^{n}$ matrices obeying the relation

$$\Gamma_{\alpha}\Gamma_{\beta} + \Gamma_{\beta}\Gamma_{\alpha} = 2\delta_{\alpha\beta}\,. \tag{4.28}$$

In this case, the spinors are represented by the column vectors on which these matrices act and their spinorial covariant derivative are given by:

$$\hat{\nabla}_{\alpha}\boldsymbol{\psi} = \partial_{\alpha}\boldsymbol{\psi} - \frac{1}{4}\omega_{\alpha}{}^{\beta\varepsilon}\Gamma_{\beta}\Gamma_{\varepsilon}\,\boldsymbol{\psi}\,, \tag{4.29}$$

with ∂_{α} denoting the partial derivative along the vector field $\boldsymbol{e}_{\alpha}$. We should observe that the Dirac matrices are unchanged under conformal tranformations, $\hat{\Gamma}_{\alpha} = \Gamma_{\alpha}$. If $\boldsymbol{\psi} \in \Gamma(SM)$ is a spinor field, let us define

$$\hat{\boldsymbol{\psi}} = \Omega^{p}\,\boldsymbol{\psi}\,, \tag{4.30}$$

with p being a constant parameter that will be conveniently chosen in the sequel. Then, using the equation (4.26), we obtain that the Dirac operator $\boldsymbol{D} = \Gamma^{\alpha}\hat{\nabla}_{\alpha}$ behaves under conformal transformations as follows

$$\begin{aligned} \hat{\boldsymbol{D}}\,\hat{\boldsymbol{\psi}} &= \hat{\boldsymbol{D}}\,(\Omega^{p}\,\boldsymbol{\psi}) \\ &= \Omega^{p-1}\,\boldsymbol{D}\boldsymbol{\psi} + \left(p + n - \frac{1}{2}\right)\Omega^{p-2}(\partial_{\alpha}\Omega)\Gamma^{\alpha}\boldsymbol{\psi}\,. \end{aligned} \tag{4.31}$$

Thus, choosing $p = -n + \frac{1}{2}$, it follows the below relations

$$\boldsymbol{D}\boldsymbol{\psi} \,=\, \Omega^{n+\frac{1}{2}}\,\hat{\boldsymbol{D}}\,\hat{\boldsymbol{\psi}} \quad , \quad \boldsymbol{\psi} \,=\, \Omega^{n-\frac{1}{2}}\,\hat{\boldsymbol{\psi}}\,. \tag{4.32}$$

It means that the monogenic equation is invariant under conformal transformations. Indeed, if $\hat{\boldsymbol{\psi}} \in \Gamma(SM)$ is a spinor field satisfying the monogenic equation on the manifold $(M, \hat{\boldsymbol{g}})$ then

$$\hat{\boldsymbol{D}}\,\hat{\boldsymbol{\psi}} \,=\, 0 \quad \Rightarrow \quad \boldsymbol{D}\boldsymbol{\psi} \,=\, 0\,, \tag{4.33}$$

since Ω is a positive definite function throughout the manifold. Thus, the problem of finding solutions of the monogenic equation in the manifold $(M, \hat{\boldsymbol{g}})$ is reduced to finding monogenic spinor fields on $(M, \boldsymbol{g})$.

In the recent years much success has been provided in the particular context of conformally flat spin manifolds, which are Riemannian manifolds with a vanishing Weyl tensor. Let $(M, \hat{\boldsymbol{g}})$ be a d-dimensional conformally flat manifolds, that is

$$\hat{g}_{\alpha\beta} \,=\, \Omega^2 \delta_{\alpha\beta} \quad , \quad \Omega \in \mathfrak{F}(M)\,, \tag{4.34}$$

in the neighborhood of a point p of $(M, \hat{\boldsymbol{g}})$. Let us present a simple construction of a solution to the monogenic equation in this manifold. Due to the conformal invariance of the monogenic equation, it is sufficient to find the monogenic spinor fields in the flat space which, in the case of Euclidean signature, is the space $\mathbb{R}^d$ generated by the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d\}$. In order to accomplish this, let us use a suitable representation for the Dirac matrices introduced in the section 2.5, but here with a slight modification. We recall that

$$\sigma_1 \,=\, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; ; \sigma_2 \,=\, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \; ; \sigma_3 \,=\, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,, \tag{4.35}$$

are the hermitian Pauli matrices and we will denote the 2×2 identity matrix by $\mathbb{I}$. Instead of spliting the representation of the Dirac matrices into those that are even or odd as Γ_{2a} or Γ_{2a-1}, let us use the following notation

$$\begin{aligned} \Gamma_a \,&=\, \underbrace{\sigma_3 \otimes \ldots \otimes}_{(a-1)\text{ times}} \sigma_1 \underbrace{\otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{(n-a)\text{ times}} \\ \Gamma_{\tilde{a}} \,&=\, \underbrace{\sigma_3 \otimes \ldots \otimes}_{(a-1)\text{ times}} \sigma_2 \underbrace{\otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{(n-a)\text{ times}}\,, \end{aligned} \tag{4.36}$$

where a and $\tilde{a}$ are indices that range from 1 to $n = d/2$. Indeed, we can easily check that the Clifford algebra given in equation (4.28) is properly satisfied by the above matrices[1]. The Euclidean Dirac operator is then represented by

$$\boldsymbol{D} \,=\, \sum_{a=1}^{n} (\Gamma_a \partial_a \,+\, \Gamma_{\tilde{a}} \partial_{\tilde{a}}) \,=\, \sum_{a=1}^{n} \underbrace{\sigma_3 \otimes \ldots \otimes}_{(a-1)\text{ times}} D_a \underbrace{\otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{(n-a)\text{ times}}\,, \tag{4.37}$$

[1]In $d = 2n + 1$, besides the $2n$ Dirac matrices Γ_a and $\Gamma_{\tilde{a}}$ we need to add one further matrix, which will be denoted by Γ_{n+1} given by $\Gamma_{n+1} = \underbrace{\sigma_3 \otimes \sigma_3 \ldots \otimes \sigma_3}_{n\text{ times}}$.

where

$$D_a = \sigma_1 \partial_a + \sigma_2 \partial_{\tilde{a}} \,,$$

is the Dirac operator on $\mathbb{R}^2$ with coordinates $\{x^a, y^a\}$. Let us assume that the spinor field $\boldsymbol{\psi}$ can be written as the following direct product of two component spinors

$$\boldsymbol{\psi} \;=\; \boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2 \otimes \ldots \otimes \boldsymbol{\psi}_n \,, \tag{4.38}$$

where each spinor $\boldsymbol{\psi}_a$ $(a = 1, 2, \ldots, n)$ has the form

$$\boldsymbol{\psi}_a \;=\; \sum_s \psi_a^{s_a} \, \boldsymbol{\xi}^{s_a} \,, \tag{4.39}$$

with their components $\psi_a^{s_a}$ $(s_a = \pm)$ depending just on the coordinates x^a and y^a, that is $\psi_a^{s_a} = \psi_a^{s_a}(x^a, y^a)$. Here, we have introduced as the basis of the spinors the column vectors

$$\xi^+ \;=\; \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad \xi^- \;=\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{4.40}$$

From equation (4.27), using (4.37) we obtain that the spinor field of expression (4.38) satisfies

$$\begin{aligned} D_1 \boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_2 \otimes \ldots \otimes \boldsymbol{\psi}_n \,+\, \sigma_3 \boldsymbol{\psi}_1 \otimes D_2 \boldsymbol{\psi}_2 \otimes \ldots \otimes \boldsymbol{\psi}_n \\ + \sigma_3 \boldsymbol{\psi}_1 \otimes \sigma_3 \boldsymbol{\psi}_2 \otimes \ldots \otimes D_n \boldsymbol{\psi}_n \;=\; 0 \,, \end{aligned} \tag{4.41}$$

which is satisfied if we choose each $\boldsymbol{\psi}_a$ such that it belongs to the kernel of the Dirac operator D_a on $\mathbb{R}^2$ and whose form can be easily obtained. It is given by

$$D_a \boldsymbol{\psi}_a = 0 \quad \Rightarrow \quad \boldsymbol{\psi}_a(x^a, y^a) \;=\; \boldsymbol{\psi}_a(x^a + i s_a y^a) \,.$$

Thus, the monogenic equation on $\mathbb{R}^d$ for a spinor fields $\boldsymbol{\psi}$ can be reduced to a monogenic equations on $\mathbb{R}^2$ for each spinor field $\boldsymbol{\psi}_a$. It gives a kind of holomorphicity conditions and allows us to construct monogenic spinor fields on flat space as tensor product of spinor fields defined on 2-dimensional flat space with the components of $\boldsymbol{\psi}$ given by holomorphic functions $\psi_a^{s_a}(x^a + i s_a y^a)$ of the plane coordinates.

The monogenic equations for spinor fields are a particular case of the Dirac equation with zero eigenvalue, massless Dirac equations. In particular, the relation (4.33) enables us to investigate the conformal invariance of the massive Dirac equation. Indeed, if $\boldsymbol{\psi}$ is a spinorial field of mass m that obeys Dirac equation in the manifold with metric $\boldsymbol{g}$, that is $\boldsymbol{D}\boldsymbol{\psi} \;=\; m\boldsymbol{\psi}$, it is straightforward see that

$$\hat{\boldsymbol{D}} \, \hat{\boldsymbol{\psi}} \;=\; \hat{m} \, \hat{\boldsymbol{\psi}} \quad , \quad \hat{m} \;=\; \Omega^{-1} \, m \,. \tag{4.42}$$

Since, generally, Ω is a non-constant function, it follows that the massive Dirac equation is not conformally invariant, whereas the massless Dirac equation is invariant under conformal transformations. Such massless equations for the Weyl spinors are known as Weyl equations. In reference [70] it is made the separation of the Neutrino Equations in a Kerr Background. The called zero-mode of the Dirac operator is a non-trivial global solution to the euclidean massless Dirac equation. In reference [65] it is made a study Dirac operator zero-modes on a torus for gauge background with uniform field strengths. In particular it is shown that under the basic translations of the torus coordinates the components of the spinor are subject to twisted periodic conditions and by a suitable choice of coordinates in the torus the zero-mode wave functions can be related to holomorphic functions of the complex torus coordinates and finally it shown that the chirality and the degeneracy of the zero-modes are uniquely determined by the gauge background and are consistent with the index theorem.

In the next section we will present the main results of this dissertation. It will be shown that the Dirac Equation coupled to a gauge field can be decoupled in even-dimensional manifolds that are the direct product of bidimensional spaces. We use the conformal invariance of the massless Dirac equation to decouple the equation of motion of a charged test field of spin 1/2 propagating in a particular black hole solution background.

4.5 Direct Product Spaces and the Separability of the Dirac Equation

Our goal in this section is to show that the Dirac equation minimally coupled to an electromagnetic field is separable in spaces that are the direct product of bidimensional spaces. In order to accomplish this, let us first fix some notations. The Greek letters from the beginning of the alphabet $(\alpha, \beta, \varepsilon)$ run from 1 to $d = 2n$ and label, as previously, the vector fields of an orthonormal frame $\{\boldsymbol{e}_\alpha\}$; lowercase Latin indices with and without tildes $(a, b, \ldots, \tilde{a}, \tilde{b}, \ldots)$ range from 1 to n and are also used to label the vector fields of an orthonormal frame $\{\boldsymbol{e}_a, \boldsymbol{e}_{\tilde{a}}\}$ in a pairwise form, which will be quite suitable to our intent, as will be clear in the sequel; the indices $(\ell, \tilde{\ell})$ run from 2 to n and serve to label the angular directions of the black hole spacetime considered here; finally, the indices $(s, s_1, s_2, \ldots)$ can take the values ± 1 and label spinorial degrees of freedom. We continue to use Einstein's summation convention, but when equal indices are in the same position, both up or both down, they should not be summed in principle, unless an explicit sign of sum is included. In what follows, we shall deal with a d-dimensional spin manifold $(M, \hat{\boldsymbol{g}})$ endowed with a metric $\hat{\boldsymbol{g}}$ of arbitrary signature.

Using the above notation, if we introduce an orthonormal frame $\{\hat{\boldsymbol{e}}_a\}$ for $\Gamma(TM)$

we have

$$\hat{\boldsymbol{g}}(\hat{\boldsymbol{e}}_\alpha, \hat{\boldsymbol{e}}_\beta) \,=\, \delta_{\alpha\beta} \quad \leftrightarrow \quad \begin{cases} \hat{\boldsymbol{g}}(\hat{\boldsymbol{e}}_a, \hat{\boldsymbol{e}}_b) &=& \delta_{ab} \\ \hat{\boldsymbol{g}}(\hat{\boldsymbol{e}}_a, \hat{\boldsymbol{e}}_{\tilde{b}}) &=& 0 \\ \hat{\boldsymbol{g}}(\hat{\boldsymbol{e}}_{\tilde{a}}, \hat{\boldsymbol{e}}_{\tilde{b}}) &=& \delta_{\tilde{a}\tilde{b}} \end{cases} . \tag{4.43}$$

The manifold $(M, \hat{\boldsymbol{g}})$ is a direct product of n bidimensional spaces which can be covered by coordinates $\{x^1, y^1, x^2, y^2, \ldots, x^n, y^n\}$ such that the line element is written as

$$d\hat{s}^2 \,=\, \sum_{a=1}^{n} d\hat{s}_a^2 \,=\, \sum_{a=1}^{n} (\hat{\boldsymbol{e}}^a\, \hat{\boldsymbol{e}}^a \,+\, \hat{\boldsymbol{e}}^{\tilde{a}}\, \hat{\boldsymbol{e}}^{\tilde{a}})\,, \tag{4.44}$$

where each 2-dimensional line elements $d\hat{s}_a^2$ and the 1-forms $\hat{\boldsymbol{e}}^a = \hat{e}_\mu^{\,a} dx^\mu$ and $\hat{\boldsymbol{e}}^{\tilde{a}} = \hat{e}_\mu^{\,\tilde{a}} dx^\mu$ should depend just on the two coordinates corresponding to their bidimensional spaces. Actually, this is our hypothesis. Note, for instance, that $d\hat{s}_1^2$, $\hat{\boldsymbol{e}}^1$ and $\hat{\boldsymbol{e}}^{\tilde{1}}$ depend just on the differentials dx^1 and dy^1 and their components should depend just on the coordinates x^1 and y^1. In such a case, the only components of the spin connection that are potentially non-vanishing are

$$\hat{\omega}_{a\tilde{a}a} \,=\, -\,\omega_{aa\tilde{a}} \quad , \quad \hat{\omega}_{\tilde{a}\tilde{a}a} \,=\, -\,\omega_{\tilde{a}a\tilde{a}}\,. \tag{4.45}$$

Thus, for example, $\omega_{a\tilde{b}a} = 0$ and $\omega_{\tilde{a}\tilde{a}b} = 0$ if $a \neq b$. Furthermore, the non-null spin connections for some index a depend just on the coordinates x^a and y^a. For instance, $\hat{\omega}_{1\tilde{1}1}$ depends just on x^1 and y^1.

In order to accomplish the **separability** of the Dirac equation, let us use the convenient representation (4.36) for the Dirac matrices. We can introduce a basis of this representation by the direct products of spinors $\boldsymbol{\xi}^s$ $(s = \pm)$ given in (4.40) which, under the action of the Pauli matrices, satisfy concisely the relations

$$\sigma_1 \boldsymbol{\xi}^s \,=\, \boldsymbol{\xi}^{-s} \quad , \quad \sigma_2 \boldsymbol{\xi}^s \,=\, i\, s\, \boldsymbol{\xi}^{-s} \quad , \quad \sigma_3 \boldsymbol{\xi}^s \,=\, s\, \boldsymbol{\xi}^s\,. \tag{4.46}$$

The basis in $d = 2n$ dimensions for the spinor space, the space on which the Dirac matrices act, is then spanned by

$$\boldsymbol{\xi}^{s_1} \otimes \boldsymbol{\xi}^{s_2} \otimes \ldots \otimes \boldsymbol{\xi}^{s_n}\,.$$

Once the base is defined, any spinor field can be expanded on this basis as

$$\hat{\boldsymbol{\psi}} \,=\, \sum_{\{s\}} \hat{\psi}^{s_1 s_2 \ldots s_n}\, \boldsymbol{\xi}^{s_1} \otimes \boldsymbol{\xi}^{s_2} \otimes \ldots \otimes \boldsymbol{\xi}^{s_n}\,, \tag{4.47}$$

where the sum over $\{s\}$ means the sum over all possible values of $\{s_1, s_2, \ldots, s_n\}$ and $\hat{\psi}^{s_1 s_2 \ldots s_n}$ stands for the components of $\hat{\boldsymbol{\psi}}$. Remember that every s_a can take two values. It means, in turn, that this sum comprises 2^n terms, which is the dimension of the spinor space in d dimensions as viewed in the section 2.4. This basis is very convenient, since the action of the Dirac matrices on the spinor fields can be easily comptuted. Indeed, using the equations (4.36), (4.46) and (4.47) that

$$\begin{aligned}\Gamma_a\hat{\boldsymbol{\psi}} &= \sum_{\{s\}}(s_1s_2\ldots s_{a-1})\hat{\psi}^{s_1s_2\ldots s_n}\,\boldsymbol{\xi}^{s_1}\otimes\boldsymbol{\xi}^{s_2}\otimes\ldots\otimes\boldsymbol{\xi}^{s_{a-1}}\otimes\boldsymbol{\xi}^{-s_a}\otimes\boldsymbol{\xi}^{s_{a+1}}\otimes\\ \ldots\quad&\otimes\boldsymbol{\xi}^{s_n} = \sum_{\{s\}}(s_1s_2\ldots s_a)\,\hat{\psi}^{s_1s_2\ldots s_{a-1}(-s_a)s_{a+1}\ldots s_n}\,\boldsymbol{\xi}^{s_1}\otimes\boldsymbol{\xi}^{s_2}\otimes\\ \ldots\quad&\otimes\boldsymbol{\xi}^{s_{a-1}}\otimes\boldsymbol{\xi}^{s_a}\otimes\boldsymbol{\xi}^{s_{a+1}}\otimes\ldots\otimes\boldsymbol{\xi}^{s_n}\,,\end{aligned}\tag{4.48}$$

where from the first to the second line we have changed the index s_a to $-s_a$, which does not change the final result, since we are summing over all values of s_a, which comprise the same list of the values of $-s_a$. Moreover, we have used that $(s_a)^2 = 1$. Analogously, we have:

$$\begin{aligned}\Gamma_{\tilde{a}}\hat{\boldsymbol{\psi}} &= \sum_{\{s\}}(s_1s_2\ldots s_{a-1})\,(is_a)\,\hat{\psi}^{s_1s_2\ldots s_n}\,\boldsymbol{\xi}^{s_1}\otimes\boldsymbol{\xi}^{s_2}\otimes\ldots\otimes\boldsymbol{\xi}^{s_{a-1}}\otimes\boldsymbol{\xi}^{-s_a}\otimes\boldsymbol{\xi}^{s_{a+1}}\otimes\\ \ldots\quad&\otimes\boldsymbol{\xi}^{s_n} = -\,i\sum_{\{s\}}(s_1s_2\ldots s_a)\,s_a\,\hat{\psi}^{s_1s_2\ldots s_{a-1}(-s_a)s_{a+1}\ldots s_n}\,\boldsymbol{\xi}^{s_1}\otimes\boldsymbol{\xi}^{s_2}\otimes\\ \ldots\quad&\otimes\boldsymbol{\xi}^{s_{a-1}}\otimes\boldsymbol{\xi}^{s_a}\otimes\boldsymbol{\xi}^{s_{a+1}}\otimes\ldots\otimes\boldsymbol{\xi}^{s_n}\,.\end{aligned}\tag{4.49}$$

Here, the Dirac operator has the same form of (4.37) just replacing the partial derivative by the covariant derivative, that is

$$\hat{\boldsymbol{D}} = \sum_{a=1}^{n}(\Gamma_a\hat{\nabla}_a + \Gamma_{\tilde{a}}\hat{\nabla}_{\tilde{a}})\,.\tag{4.50}$$

All that was seen above are necessary tools to attack our initial problem of separating the general equation

$$\left[\hat{\boldsymbol{D}} - (\Gamma^a\hat{A}_a + \Gamma^{\tilde{a}}\hat{A}_{\hat{a}})\right]\hat{\boldsymbol{\psi}} = \hat{m}\,\hat{\boldsymbol{\psi}}\tag{4.51}$$

in its 2-dimensional blocks, where $\hat{A}_a, \hat{A}_{\hat{a}}$ and $\hat{m}$ are arbitrary functions of the coordinates. In order to perform this separation, for the spinor field (4.47) let us use the ansatz take the following separable form

$$\hat{\psi}^{s_1s_2\ldots s_n} = \hat{\psi}_1^{s_1}(x^1,y^1)\,\hat{\psi}_2^{s_2}(x^2,y^2)\,\ldots\,\hat{\psi}_n^{s_n}(x^n,y^n)\,.\tag{4.52}$$

From this hypothesis and using the equations (4.48) and (4.49) it follows that the equation (4.51) is given by

$$\begin{aligned}
&\sum_{a=1}^{n}\sum_{\{s\}} (s_1 s_2 \ldots s_a)\, \hat{\psi}_1^{s_1} \ldots \hat{\psi}_{a-1}^{s_{a-1}}\, \hat{\psi}_{a+1}^{s_{a+1}} \ldots \hat{\psi}_n^{s_n} \\
&\left[i\, s_a \left(\hat{\partial}_a + \frac{1}{2}\,\hat{\omega}_{\tilde{a}a\tilde{a}} - \hat{A}_a\right) + \left(\hat{\partial}_{\tilde{a}} + \frac{1}{2}\,\hat{\omega}_{a\tilde{a}a} - \hat{A}_a\right)\right] \hat{\psi}_a^{(-s_a)}\, \boldsymbol{\xi}^{s_1} \otimes \boldsymbol{\xi}^{s_2} \otimes \ldots \otimes \boldsymbol{\xi}^{s_n} \\
= \;& i\,\hat{m} \sum_{\{s\}} \hat{\psi}_1^{s_1}\, \hat{\psi}_2^{s_2}\, \ldots\, \hat{\psi}_n^{s_n}\, \boldsymbol{\xi}^{s_1} \otimes \boldsymbol{\xi}^{s_2} \otimes \ldots \otimes \boldsymbol{\xi}^{s_n}\,, \qquad (4.53)
\end{aligned}$$

where $\hat{\partial}_a$ and $\hat{\partial}_{\tilde{a}}$ are the derivatives along the vector fields $\hat{\boldsymbol{e}}_a$ and $\hat{\boldsymbol{e}}_{\tilde{a}}$, respectively. Let us write this latter equation in a more convenient form. Factorizing the components $\hat{\psi}_1^{s_1}\, \hat{\psi}_2^{s_2}\, \ldots\, \hat{\psi}_n^{s_n}$ of the spinor, we have

$$\begin{aligned}
&\sum_{\{s\}} \left(\sum_{a=1}^{n} (s_1 s_2 \ldots s_a)\, \frac{1}{\hat{\psi}_a^{s_a}} \left[i\, s_a \left(\hat{\partial}_a + \frac{1}{2}\,\hat{\omega}_{\tilde{a}a\tilde{a}} \,-\, \hat{A}_a\right) + \left(\hat{\partial}_{\tilde{a}} + \frac{1}{2}\,\hat{\omega}_{a\tilde{a}a} - \hat{A}_a\right)\right] \hat{\psi}_a^{(-s_a)}\right. \\
&\left. -\, i\,\hat{m}\, \hat{\psi}_1^{s_1}\, \hat{\psi}_2^{s_2}\, \ldots\, \hat{\psi}_n^{s_n}\, \boldsymbol{\xi}^{s_1} \otimes \boldsymbol{\xi}^{s_2} \otimes \ldots \otimes \boldsymbol{\xi}^{s_n} = 0 \right. \qquad (4.54)
\end{aligned}$$

We can thus conclude that

$$\begin{aligned}
&\sum_{a=1}^{n} (s_1 s_2 \ldots s_a)\, \frac{1}{\hat{\psi}_a^{s_a}} \left[i\, s_a \left(\hat{\partial}_a + \frac{1}{2}\,\hat{\omega}_{\tilde{a}a\tilde{a}} - \hat{A}_a\right) + \left(\hat{\partial}_{\tilde{a}} + \frac{1}{2}\,\hat{\omega}_{a\tilde{a}a} - \hat{A}_a\right)\right] \hat{\psi}_a^{(-s_a)} \\
= \;& i\,\hat{m}\,. \qquad (4.55)
\end{aligned}$$

In order for the latter equation to be separable in blocks depending only on the coordinates $\{x_a, y_a\}$ for each value of a, we assume that the functions $\hat{A}_a$ and $\hat{A}_{\tilde{a}}$ must depend only on the two coordinates $\{x^a, y^a\}$ and using this assumption, since now the left hand side depend only on these pairs of coordinates the function $\hat{m}$ must be a sum over a of functions depending also on these pairs of coordinates. We can summarize these results by the relations

$$\hat{A}_a = \hat{A}_a(x^a, y^a) \quad , \quad \hat{A}_{\tilde{a}} = \hat{A}_{\tilde{a}}(x^a, y^a) \quad , \quad \hat{m} = \sum_{a=1}^{n} \hat{m}_a(x^a, y^a)\,. \qquad (4.56)$$

These assumptions lead us to the following equation:

$$\sum_{a=1}^{n} \left[(s_1 s_2 \ldots s_a)\, \frac{1}{\hat{\psi}_a^{s_a}} D_a^{s_a} \hat{\psi}_a^{(-s_a)} \,-\, i\,\hat{m}_a\right] \,=\, 0\,, \qquad (4.57)$$

where the operator $D_a^{s_a}$ is defined by

$$D_a^{s_a} \,=\, i\, s_a \left(\hat{\partial}_a + \frac{1}{2}\,\hat{\omega}_{\tilde{a}a\tilde{a}} - \hat{A}_a\right) + \left(\hat{\partial}_{\tilde{a}} + \frac{1}{2}\,\hat{\omega}_{a\tilde{a}a} - \hat{A}_a\right)\,. \qquad (4.58)$$

Note that each term in the sum over a depends just on the two coordinates $\{x^a, y^a\}$. In order to the sum of functions depending on distinct variables to be zero, is thus necessary that each term of these sum be a constant, here called of separation constant, with the sum of the constants being null. However, it is worth noting that the separation constants can depend on the choice of $\{s\} \equiv \{s_1, s_2, \ldots, s_n\}$. Indeed, the equation (4.57) provides not only one equation but rather a total of 2^n independent equations, since for each choice of $\{s\} \equiv \{s_1, s_2, \ldots, s_n\}$ we have one equation. Let us denote this separation constant conveniently by $i\eta_a^{\{s\}}$. Then, we find the following set of coupled first orders for $\hat{\psi}_a^{s_a}$ and $\hat{\psi}_a^{(-s_a)}$:

$$(s_1 s_2 \ldots s_a)\, D_a^{s_a} \hat{\psi}_a^{(-s_a)} \;=\; i\,(\hat{m}_a + i\eta_a^{\{s\}})\,\hat{\psi}_a^{s_a} \quad , \quad \sum_{a=1}^{n} \eta_a^{\{s\}} = 0\,. \tag{4.59}$$

For each of these equations we can have different separation constants. These equations enable us to integrate the fields $\hat{\psi}_a^{s_a}$ and, therefore, find the solutions for the generalized Dirac equation (4.51). We obtain thus that in the manifold with line element (4.44) the generalized Dirac equation (4.51) reduces to pair of first order differential equations. However, although these equations are first order differential equations, they are coupled in pairs, namely the equations involving the field $\hat{\psi}_a^{s_a}$ have $\hat{\psi}_a^{(-s_a)}$ as source and vice-versa. Therefore, we can eliminate $\hat{\psi}_a^{s_a}$ or $\hat{\psi}_a^{(-s_a)}$ from the of equations (4.59) to obtain a decoupled second order differential equation for each component $\hat{\psi}_a^{s_a}$, thus achieving the separability that we were looking for.

The second equation of (4.59) are constraints which the separation constants must obey. Let us solve these constraints. What makes our task nontrivial is that since each one of the n spinorial indices $s_a\,(a = 1, 2, \ldots, n)$ can take two values, the collective "index" $\{s\}$ can take 2^n values, it follows that the equation

$$\sum_{a=1}^{n} \eta_a^{\{s\}} = 0 \tag{4.60}$$

comprise 2^n constraints. But, it is worth noting that $\eta_a^{\{s\}}$ cannot depend of $s_{a+1}, s_{a+2}, \ldots, s_n$, since the first equation of (4.59) by consistency is independent on these indices. In particular, by writing the quoted equation in the form

$$i\eta_a^{\{s\}} \;=\; (s_1 s_2 \ldots s_a)\frac{1}{\hat{\psi}_a^{s_a}}\, D_a^{s_a} \hat{\psi}_a^{(-s_a)} \;-\; i\,\hat{m}_a \tag{4.61}$$

we can see that $\eta_1^{\{s\}}$ depends just on s_1, so that we can write it as

$$\eta_1^{\{s\}} \;=\; s_1\,\kappa_1^{s_1}\,, \tag{4.62}$$

where $\kappa_1^{s_1}$ is a pair of constants that depends just on s_1. Now, multiplying the equation (4.61) by $s_1 s_2 \ldots s_a$ and using that $s_a^2 = 1$ we can write

$$\frac{1}{\hat{\psi}_a^{s_a}}\, D_a^{s_a} \hat{\psi}_a^{(-s_a)} \;=\; i\,(s_1 s_2 \ldots s_a)(\hat{m}_a + \eta_a^{\{s\}})\,\hat{\psi}_a^{s_a}\,. \tag{4.63}$$

Note that the right hand side depends on $s_1, s_2, \ldots, s_a$ while the left side depends only on s_a. In order to the right hand side of this equation, likewise, depend just on s_a we must have

$$\hat{m}_a + \eta_a^{\{s\}} \;=\; (s_1 s_2 \ldots s_a)\, \kappa_a^{s_a} \quad , \quad a \geq 2\,, \tag{4.64}$$

where differently of $\kappa_1^{s_1}$, here $\kappa_a^{s_a}$ is a pair of parameters in principle not constant that depends just on s_a for each a to determine in the sequel. Since $\eta_a^{\{s\}}$ are constants, taking the derivative of both sides of the latter equation, we have

$$\partial_\mu \hat{m}_a \;=\; (s_1 s_2 \ldots s_a)\, \partial_\mu \kappa_a^{s_a}\,. \tag{4.65}$$

Since the left hand side of the latter equation does not depend on the elements of the set $\{s\}$, it follows that $\partial_\mu \kappa_a^{s_a}$ must vanish for the equation to be consistent. This, in turn, implies that $\hat{m}_a$ should be constant for $a \geq 2$. But, if $\hat{m}_2(x^2, y^2), \hat{m}_3(x^3, y^3), \ldots, \hat{m}_n(x^n, y^n)$ are constants we can, without loss of generality, make all of them zero and absorb these constants in $\hat{m}_1(x^1, y^1)$. Therefore, we can say that a consistent separability process requires that

$$\hat{m}_2 \;=\; \hat{m}_3 \;=\; \ldots \hat{m}_n \;=\; 0\,. \tag{4.66}$$

Once assumed the previous conditions, the equations (4.62) and (4.64) immediately lead to

$$\eta_a^{\{s\}} \;=\; (s_1 s_2 \ldots s_a)\, \kappa_a^{s_a}\,, \tag{4.67}$$

where now $\kappa_a^{s_a}$ is a pair of constants as $\kappa_1^{s_1}$. Since the sum of the constants $\eta_a^{\{s\}}$ vasinhes, we are left with the following equation

$$\sum_{a=1}^{n} (s_1 s_2 \ldots s_a)\, \kappa_a^{s_a} \;=\; 0\,, \tag{4.68}$$

which must hold for all possible choices of $\{s\}$. We thus wish know who are the constants $\kappa_a^{s_a}$. In order to perform this we need manipulate this equation to solve such constraint. From now on, our task is restricted to identifying on both sides of the equation their corresponding dependencies with respect to the elements of the set $\{s\}$. Indeed, isolating $\kappa_1^{s_1}$ we obtain that

$$\kappa_1^{s_1} \;=\; -\sum_{a=2}^{n} (s_2 \ldots s_a)\, \kappa_a^{s_a}\,.$$

On the left hand side of the equation we have the term corresponding to the value of a equal to 1 which depends just on s_1 and on the right hand side a sum of terms that, in principle, may be dependent on s_1. However, since the sum start counting from a equal to 2 forward, none of the terms in the sum depend on s_1. We can conclude that this sum is constant, that is

$$\sum_{a=2}^{n} (s_2 \dots s_a)\, \kappa_a^{s_a} \,=\, c_1 \quad , \quad \kappa_1^{s_1} \,=\, -\, c_1 \,,$$

where c_1 is a constant that does not depend on $\{s\}$. Following the same reasoning, let us write the latter equation as follows

$$\kappa_2^{s_2} \,-\, s_2\, c_1 \,=\, -\sum_{a=3}^{n} (s_3 \dots s_a)\, \kappa_a^{s_a} \,.$$

Noting that the left hand side of the above equation depends just on s_2 and that the sum of the terms on the right hand side clearly do not depend on s_2, we thus can conclude that

$$\sum_{a=3}^{n} (s_3 \dots s_a)\, \kappa_a^{s_a} \,=\, c_2 \quad , \quad \kappa_2^{s_2} \,=\, s_2\, c_1 \,-\, c_2 \,,$$

where c_2 is a constant that does not depend on $\{s\}$. We can continue with this same procedure until reaching the term $\kappa_n^{s_n}$. Eventually, we are left with the following final equation

$$\kappa_a^{s_a} \,=\, s_a c_{a-1} \,-\, c_a \quad , \quad c_0 \,=\, c_n \,=0 \,, \tag{4.69}$$

where $c_1, c_2, \dots, c_{n-1}$ are arbitrary constants. Hence, this problem admits $(n-1)$ constants of separation. The latter equation is the general solution that solve the constraint (4.68). With the results obtained on the equations (4.66), (4.67) and (4.69), we can use them in the first equation of (4.59) to find, finally, the following set of coupled first order differential equations:

$$\begin{aligned} D_1^{s_1}\, \hat{\psi}_1^{(-s_1)} \,&=\, i\,(s_1\, \hat{m}_1 - c_1)\, \hat{\psi}_1^{s_1} \\ D_a^{s_a}\, \hat{\psi}_a^{(-s_a)} \,&=\, i\,(s_a\, c_{a-1} - c_a)\, \hat{\psi}_a^{s_a} \quad , \quad a \geq 2 \,. \end{aligned} \tag{4.70}$$

We have thus reduced the solution of the generalized Dirac equation (4.51) in the manifold with line element (4.44) n pairs of first order differential equations. Eliminating $\hat{\psi}_a^{s_a}$ or $\hat{\psi}_a^{(-s_a)}$ gives us a second order equation for $\hat{\psi}_a^{s_a}$ and the general solution can be expressed as a linear combination of the of the particular solutions belonging to the different values of $\{c_1, c_2, \dots, c_{n-1}\}$ which can only take discrete values for appropriate boundary conditions. In this sense, the constants $\{c_1, c_2, \dots, c_{n-1}\}$ can be viewed as eigenvalues and determined by the condition of regularity of the solutions $\hat{\psi}_a^{s_a}$.

In the next section, we shall use these results to separate the Dirac equation in some black hole spacetimes

4.6 Black Hole Spacetimes

The study of scalar fields, spin 1/2 fields and gauge fields (abelian and non-abelian) propagating in curved spacetimes plays a central role on the study of General relativity and any other theory of gravity. The main reason is that besides the detection of gravitational radiation and observation of the direct interaction between objects via gravitation, the most natural and simple way to probe the gravitational field permeating our spacetime is by letting other fields interact with it. In this section we shall use the previous results to separate the Dirac equation corresponding to a massive and electrically charged field of spin 1/2 in the background of black holes described in [78], which is a static black hole whose horizon have topology $\mathbb{R} \times S^2 \times \ldots \times S^2$. These black hole solutions possessing electric and magnetic charge, have been also obtained in references [87] and [86]. These are spacetimes whose line element in even dimensions $d = 2n$ is

$$ds^2 \,=\, -f(r)^2\, dt^2 \,+\, \frac{dr^2}{f(r)^2} \,+\, r^2 \sum_{\ell=2}^{n} (d\theta_\ell^2 \,+\, \sin^2\theta_\ell\, d\phi_\ell^2)\,, \tag{4.71}$$

where $f = f(r)$ is a function of the coordinate r as follows

$$f(r) = \sqrt{\frac{1}{d-3} \,+\, \frac{2\,M}{r^{\,d-3}} \,+\, \frac{Q_e^2(d-3)}{2(d-2)r^{\,2(d-3)}} \,-\, \frac{Q_m^2}{4(d-5)r^{\,2}} \,-\, \frac{\Lambda\, r^{\,2}}{d-1}}\,, \tag{4.72}$$

with M, Q_e and Q_m being the mass, the arbitrary electric and magnetic charges, respectively, of the black hole and Λ the cosmological constant. The details of this solution can be consulted in reference [78]. A suitable orthonormal frame of 1-forms for such spacetime is the following:

$$\boldsymbol{e}^1 = i\, f(r)\, dt \quad , \quad \boldsymbol{e}^{\tilde{1}} = \frac{1}{f(r)}\, dr \quad , \quad \boldsymbol{e}^{\ell} = r \sin\theta_\ell\, d\phi_\ell \quad , \quad \boldsymbol{e}^{\tilde{\ell}} = r\, d\theta_\ell\,, \tag{4.73}$$

where the index l ranges from 2 to n. In this frame, the line element is given by

$$ds^2 \,=\, \sum_{a=1}^{n} (\boldsymbol{e}^a\, \boldsymbol{e}^a \,+\, \boldsymbol{e}^{\tilde{a}}\, \boldsymbol{e}^{\tilde{a}})\,. \tag{4.74}$$

This spacetime is the solution of Einstein-Maxwell equations with a cosmological constant Λ and electromagnetic field $\boldsymbol{\mathcal{F}} = d\boldsymbol{A}$, where the gauge field $\boldsymbol{A}$ given by

$$\boldsymbol{A} = \frac{Q_e}{r^{d-3}}\, dt \,+\, Q_m \sum_{\ell=2}^{n} \cos\theta_\ell\, d\phi_\ell\,,$$

which, in the orthonormal frame, can be written as

$$\boldsymbol{A} = A_a\, \boldsymbol{e}^a \,+\, A_{\tilde{a}}\, \boldsymbol{e}^{\tilde{a}} = A_1\, \boldsymbol{e}^1 \,+\, \sum_{\ell=2}^{n} A_\ell\, \boldsymbol{e}^\ell\,, \tag{4.75}$$

where

$$A_1 = -\frac{i\,Q_e}{f(r)\,r^{\,d-3}} \quad , \quad A_\ell = \frac{Q_m}{r}\cot\theta_\ell \quad , \quad A_{\tilde{a}} = 0\,. \tag{4.76}$$

The Dirac equation is written in the form

$$\boldsymbol{D}\,\boldsymbol{\psi} = m\,\boldsymbol{\psi}\,, \tag{4.77}$$

where $\boldsymbol{D} = \Gamma^\alpha\nabla_\alpha$ is the dirac operator. The field $\boldsymbol{\psi}$ of spin 1/2 with electric charge q and mass m in this spacetime and minimally coupled to the electromagnetic field obeys the following version of the Dirac equation

$$(\boldsymbol{D} - i\,q\boldsymbol{\mathcal{A}})\,\boldsymbol{\psi} = m\,\boldsymbol{\psi}\,, \tag{4.78}$$

where $\boldsymbol{\mathcal{A}} = A_\alpha\Gamma^\alpha$ is the representation of the gauge field. Using that $A_{\tilde{a}} = 0$, it follows that the above equation is written as

$$\boldsymbol{D}\,\boldsymbol{\psi} = (m + i\,qA_a\Gamma^a)\,\boldsymbol{\psi}\,. \tag{4.79}$$

Our goal in this section is integrate this equation. Making a connection with the previous section, we recall that an equation analogous to this has been separated when the space is a direct product of bidimensional spaces. Clearly this is not the case, since in front of the angular part of the black hole line element (4.71) there is the multiplicative factor r^2. However, we can factor out the function r^2 in the line element (4.71) and define $d\hat{s}^2$ that is conformally related to our initial line element

$$d\hat{s}^2 = \frac{ds^2}{r^2}\,,$$

where

$$d\hat{s}^2 = -\frac{f^2}{r^2}\,dt^2 + \frac{dr^2}{(rf)^2} + \sum_{\ell=2}^{n}(d\theta_\ell^2 + \sin^2\theta_\ell\,d\phi_\ell^2) = \sum_{a=1}^{n} d\hat{s}_a^{\,2}\,, \tag{4.80}$$

with

$$d\hat{s}_1^2 = -\frac{f^2}{r^2}\,dt^2 + \frac{dr^2}{(rf)^2} \quad , \quad d\hat{s}_\ell^2 = \sum_{\ell=2}^{n}(d\theta_\ell^2 + \sin^2\theta_\ell\,d\phi_\ell^2)\,.$$

The conformally transformed space with line element $d\hat{s}^2$ is a direct product of bidimensional spaces. It is worth noting that $d\hat{s}_1^2$ depends just on the coordinates t and r, while $d\hat{s}_\ell^2$ depends on the angular coordinates θ_ℓ and ϕ_ℓ. A suitable orthonormal frame of 1-form for this space is given by

$$\hat{\boldsymbol{e}}^1 = \frac{if}{r}\,dt \quad , \quad \hat{\boldsymbol{e}}^{\tilde{1}} = \frac{1}{rf}\,dr \quad , \quad \hat{\boldsymbol{e}}^{\,\ell} = \sin\theta_\ell\,d\phi_\ell \quad , \quad \hat{\boldsymbol{e}}^{\,\tilde{\ell}} = d\theta_\ell\,. \tag{4.81}$$

In this case, the only non-vanishing components of the connection $\hat{\omega}_{abc}$ in the frame $\{\hat{\boldsymbol{e}}_a\}$ are

$$\hat{\omega}_{1\tilde{1}1} \,=\, -\,\hat{\omega}_{11\tilde{1}} = r\,f' - f \quad , \quad \hat{\omega}_{\ell\tilde{\ell}\ell} \,=\, -\,\hat{\omega}_{\ell\ell\tilde{\ell}} = \cot\theta_\ell\,, \tag{4.82}$$

where f' stands for the derivative of f with respect to its variable r. The conformal transformation of the Dirac operator given by the equation (4.42) enable us relate the quantities concerning to the spacetime with line elements ds^2 to the quantities defined on $d\hat{s}^2$. Indeed, we can write the field equation (4.79) in terms of an equation in the space with line element $d\hat{s}^2$, so that the separability results of the previous section can be fully used. In particular, we recall that the relevant conformal transformations are the following

$$\hat{\boldsymbol{D}}\,\hat{\boldsymbol{\psi}} \,=\, \Omega^{-(n+\frac{1}{2})}\,\boldsymbol{D}\boldsymbol{\psi} \quad , \quad \hat{\boldsymbol{\psi}} \,=\, \Omega^{\frac{1}{2}-n}\,\boldsymbol{\psi} \quad , \quad \hat{m} \,=\, \Omega^{-1}\,m \quad \text{with} \quad \Omega = r^{-1}\,.$$

From the latter equation follows that the field equation (4.79) can be written as

$$\hat{\boldsymbol{D}}\,\hat{\boldsymbol{\psi}} = \Omega^{-1}(m + i\,qA_a\Gamma^a)\,\hat{\boldsymbol{\psi}} \tag{4.83}$$

in its turn, defining

$$\hat{m} = \hat{m}_1 = r\,m \quad , \quad \hat{A}_a = i\,q\,rA_a \quad , \quad \hat{A}_{\tilde{a}} = 0\,, \tag{4.84}$$

we are left with the following equation

$$(\,\hat{\boldsymbol{D}} \,-\, \Gamma^a\hat{A}_a\,)\,\hat{\boldsymbol{\psi}} = \hat{m}\,\hat{\boldsymbol{\psi}}\,, \tag{4.85}$$

which is exactly the form of equation studied in previous section and we have been able so separate. Moreover, and foremost, defining the coordinates

$$x^1 = t \quad , \quad y^1 = r \quad , \quad x^\ell = \phi_\ell \quad , \quad y^\ell = \theta_\ell\,, \tag{4.86}$$

it follows that the function $\hat{m}$ and the gauge field are exactly of the form necessary to attain separability, namely the constraints (4.56) and (4.66) are obeyed. In particular, the solutions of the equation (4.79) in the black hole background is given by

$$\boldsymbol{\psi} \,=\, r^{\,(\frac{1}{2}-n)}\sum_{\{s\}}\psi_1^{s_1}(t,r)\,\psi_2^{s_2}(\phi_2,\theta_2)\,\ldots\,\psi_n^{s_n}(\phi_n,\theta_n)\,\boldsymbol{\xi}^{s_1}\otimes\boldsymbol{\xi}^{s_2}\otimes\ldots\otimes\boldsymbol{\xi}^{s_n}\,. \tag{4.87}$$

From equations (4.58), (4.70), (4.76) and (4.82) - (4.84), it follows that the functions $\psi_a^{s_a}$ must be solutions of the following differential equations

$$\left[i\,s_1\left(\frac{r}{i\,f}\,\partial_t - \frac{q\,Q_e}{f r^{\,d-4}}\right) + \left(r\,f\,\partial_r + \frac{1}{2}(r\,f' - f)\right)\right]\psi_1^{(-s_1)} = i\,(s_1\,r\,m - c_1)\,\psi_1^{s_1}$$

$$\left[i\,s_\ell\left(\frac{1}{\sin\theta_\ell}\,\partial_{\phi_\ell} - i\,q\,Q_m\cot\theta_\ell\right) + \left(\partial_{\theta_\ell} + \frac{1}{2}\cot\theta_\ell\right)\right]\psi_\ell^{(-s_\ell)} = i\,(s_\ell\,c_{\ell-1} - c_\ell)\,\psi_\ell^{s_\ell}\,, \tag{4.88}$$

where we recall that the constants of separation $c_1, c_2, \ldots, c_{n-1}$ take discrete values once boundary conditions and regularity requirements are imposed and the constant c_n is zero. It is fruitful noting that the coefficients of the latter equation are independents of the coordinate t and ϕ_ℓ, therefore both the metrics $\boldsymbol{g}$ and $\hat{\boldsymbol{g}}$ possess the Killing vector fields ∂_t and ∂_{ϕ_ℓ}. This stems from the fact that the coordinates t and ϕ_ℓ are cyclic coordinates of this metric. It is thus convenient to decompose the dependence of the fields $\psi_a^{s_a}$ on these coordinates in the Fourier basis, namely,

$$\psi_1^{s_1}(t,r) = e^{i\omega t}\,\Psi_1^{s_1}(r) \quad , \quad \psi_\ell^{s_\ell}(\phi_\ell,\theta_\ell) = e^{i\omega_\ell\phi_\ell}\,\Psi_\ell^{s_\ell}(\theta_\ell)\,. \tag{4.89}$$

The final general solution for the field $\boldsymbol{\psi}$ must, then, include a "sum" over all values of the Fourier frequencies ω and ω_ℓ with arbitrary Fourier coefficients. While ω can be interpreted as related to the energy of the field, ω_ℓ are related to angular momentum. Note that in order to avoid conical singularities in the spacetime, the coordinates ϕ_ℓ must have period 2π, namely ϕ_ℓ and $\phi_\ell + 2\pi$ should be identified [88]. As it is well-known, a spin 1/2 field changes its sign after a 2π rotation, which implies that the angular frequencies ω_ℓ must be half-integers, see [88] for more details:

$$\omega_\ell = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots\,. \tag{4.90}$$

Finally, inserting the decomposition (4.89) into equation (4.88), we end up with the following pairwise coupled system of differential equations:

$$\begin{aligned}
&\left[r\,f\,\frac{d}{dr} + \frac{1}{2}(r\,f' - f) + i\,s_1\left(\frac{\omega\,r}{f} - \frac{q\,Q_e}{f r^{\,d-4}}\right) + \right]\Psi_1^{(-s_1)} = i\,(s_1\,r\,m - c_1)\,\Psi_1^{s_1}\\
&\left[\frac{d}{d\theta_\ell} + \frac{1}{2}\cot\theta_\ell - \,s_\ell\left(\frac{\omega_\ell}{\sin\theta_\ell} - \,q\,Q_m\cot\theta_\ell +\right)\right]\Psi_\ell^{(-s_\ell)} = i\,(s_\ell\,c_{\ell-1} - c_\ell)\,\Psi_\ell^{s_\ell}\ .
\end{aligned}\tag{4.91}$$

4.6.1 The angular part of Dirac's Equation

Since we have been able so separate the generalized Dirac equation into radial and angular coordinates, we now shall investigate a little further these equations. Let us start with the angular part of the equations, namely the equations for $\Psi_\ell^{s_\ell}$. One can make a simplification on these equations by performing a field redefinition along with a redefinition of the separation constants, as we show in the sequel. Instead of using the $(n-1)$ separation constants $c_1, c_2, \ldots, c_{n-1}$, we shall use the constants $\lambda_2, \lambda_3, \ldots, \lambda_n$, defined by

$$\lambda_\ell \equiv \sqrt{c_{\ell-1}^2 - c_\ell^2}\,, \tag{4.92}$$

where we recall that $c_n = 0$, by definition. Inverting these relations, we find that the old constants can be written in terms of the new constants as follows:

$$c_{\ell-1} = \sqrt{\lambda_\ell^2 + \lambda_{\ell+1}^2 + \ldots + \lambda_n^2}\,. \tag{4.93}$$

Then, defining the parameter

$$\zeta_\ell = \text{arctanh}\left(\frac{c_\ell}{c_{\ell-1}}\right) ,$$

we find that

$$c_{\ell-1} = \lambda_\ell \cosh\zeta_\ell \quad , \quad c_\ell = \lambda_\ell \sinh\zeta_\ell \, ,$$

so that the following relation holds:

$$s_\ell \, c_{\ell-1} - c_\ell = s_\ell \, \lambda_\ell \, e^{-\,s_\ell \zeta_\ell} \, .$$

Thus, performing the field redefinition given by

$$\Psi_\ell^{s_\ell}(\theta) = e^{\,s_\ell \, \zeta_\ell/2} \, \Phi_\ell^{s_\ell}(\theta) \, , \tag{4.94}$$

it turns out that the angular part of (4.91) can be written in a simpler way in terms of the fields $\Phi_\ell^{s_\ell}(\theta)$. Indeed, using the latter equation on the second equation of (4.91) we find that

$$\left[\frac{d}{d\theta_\ell} + \frac{1}{2}\cot\theta_\ell - \, s_\ell\left(\frac{\omega_\ell}{\sin\theta_\ell} - \, q\, Q_m \cot\theta_\ell\right)\right] \Phi_\ell^{(-s_\ell)} = i\, s_\ell \, \lambda_\ell \, \Phi_\ell^{s_\ell} \, . \tag{4.95}$$

Although it may seem that we did not achieve much simplification by the redefinition of the fields and separation constants, it turns out that in the case in which the black hole has vanishing magnetic charge, $Q_m = 0$, these equations reduce to

$$\left[\frac{d}{d\theta_\ell} + \frac{1}{2}\cot\theta_\ell - \frac{s_\ell \, \omega_\ell}{\sin\theta_\ell}\right] \Phi_\ell^{(-s_\ell)} = i\, s_\ell \, \lambda_\ell \, \Phi_\ell^{s_\ell} \, . \tag{4.96}$$

The above equation is exactly the Dirac equation on the 2-sphere S^2, that is a eigenvalue equation $\boldsymbol{D}_{S^2}\boldsymbol{\Phi} = i\,\lambda\,\boldsymbol{\Phi}$, where by $\boldsymbol{D}_{S^2}$ we mean the Dirac operator on the 2-sphere and $\boldsymbol{\Phi}$ a 2-component spinor. Indeed, using the frame $\boldsymbol{e}^1 = \sin\theta d\phi$ $\boldsymbol{e}^2 = d\theta$ along with the Dirac matrices $\gamma^1 = \sigma_1$ and $\gamma^2 = \sigma_2$ one can easily derive the latter equation. The problem of finding eigenstates of Dirac operator on the sphere S^2 is a well posed mathematical problem that can be solved exactly. The solution $\boldsymbol{\Phi}$ has its components written in terms of Jacobi polynomials [88, 90] and geometrically, these solutions can be understood in terms of the Wigner elements of the group $Spin(\mathbb{R}^3)$, that give rise to the so-called spin weighted spherical harmonics [93, 94], which are tensorial generalizations of the spherical harmonics. Moreover, not all values are allowed for λ_ℓ. Indeed, once regularity requirements are impose on 2-sphere, λ_ℓ must fulfill the condition to be non-zero integers

$$\lambda_\ell = \pm 1, \pm 2, \pm 3, \dots \, . \tag{4.97}$$

Solutions with non-integer eigenvalues are not well-defined on the whole sphere, while a vanishing eigenvalue is forbidden by the Lichnerowicz theorem [92], since the sphere is a compact manifold with positive curvature.

Regarding the general case in which the black hole magnetic charge is non-vanishing, $Q_m \neq 0$, we have tried to make a redefinition of the fields $\Phi_\ell^{s_\ell}$ by means of a general linear combination of the fields Φ_ℓ^{+} and Φ_ℓ^{-}, with non-constant coefficients, in order to convert equation (4.95) into the eigenvalue equation $\boldsymbol{D}_{S^2}\boldsymbol{\Phi} = i\,\lambda\,\boldsymbol{\Phi}$. However, it turns out that the coefficients of the linear combination must obey fourth-order differential equations, whose solutions seem to be quite dificult to attain analytically. In spite of this, we can make an important progress regarding the system of equations (4.95) by decoupling the fields Φ_ℓ^{+} and Φ_ℓ^{-}, which, after all, is our goal at this section. The final result is that the fields $\Phi_\ell^{s_\ell}$ satisfy the following second order differential equation:

$$\frac{1}{\sin\theta_\ell}\frac{d}{d\theta_\ell}\left(\sin\theta_\ell\frac{d\Phi_\ell^{s_\ell}}{d\theta_\ell}\right)+$$
$$\left[\frac{(1+2qQ_m)\,\omega_\ell\cos\theta_\ell}{\sin^2\theta_\ell}-\frac{1+2qQ_m+2\omega_\ell^2}{2\sin^2\theta_\ell}+\frac{(1-4q^2Q_m^2)\cos^2\theta_\ell}{4\cos^2\theta_\ell}-\lambda_\ell^2\right]\Phi_\ell^{s_\ell}=0\,. \tag{4.98}$$

It is worth stressing that the latter equation must be supplemented by the requirement of regularity of the fields $\Phi_\ell^{s_\ell}$ at the points $\theta = 0$ and $\theta = \pi$, where our coordinate system breaks down. These regularity conditions transform the task of solving the latter equation in a Sturm-Liouville problem, so that the possible values assumed by the separation constants λ_l form a discrete set. Since the case $Q_m = 0$ in equation (4.98) has a known solution, as described above, it follows that we can look for solutions for the case $Q_m \neq 0$ by means of perturbation methods, with Q_m being the perturbation parameter. Indeed, in the celebrated paper [74], a similar path has been taken by Press and Teukolsky in order find the solutions and their eigenvalues for the angular part of the equations of motion for fields with arbitrary spin on Kerr spacetime, in which case the angular momentum of the black hole was the order parameter. In this respect, see also the reference [96].

4.6.2 The radial part of Dirac's Equation

Now, we shall investigate a little further about the pair of radial equations in expression (4.91). In order to acomplish to solve such equations we should first decouple the fields Ψ_l^{+} and Ψ_l^{-}. The form of radial equation in (4.91) suggests that instead of the factors inside of the brackets we define the following functions of the coordinate r,

$$\begin{aligned} B_{s_1}(r) &= \frac{1}{r\,f}\left[\frac{1}{2}(r\,f'-f)-i\,s_1\left(\frac{\omega\,r}{f}-\frac{q\,Q_e}{f r^{\,d-4}}\right)\right] \\ C_{s_1}(r) &= -\frac{i}{r\,f}\,(s_1\,r\,m+c_1)\,, \end{aligned}$$

in terms of which the radial equation in (4.91) can be written as

$$\frac{d\Psi_1^{s_1}}{dr} = -B_{s_1}\,\Psi_1^{s_1} + C_{s_1}\,\Psi_1^{-s_1}\,. \tag{4.99}$$

Then, deriving this equation with respect to r and eliminating $\Psi_1^{-s_1}$ of the above equation to obtain

$$\Psi_1^{-s_1} = \frac{1}{C_{s_1}}\left(\frac{d\Psi_1^{s_1}}{dr} + B_{s_1}\,\Psi_1^{s_1}\right)\,,$$

using this, we eventually arrive at the following expression:

$$\frac{d^2\Psi_1^{s_1}}{dr^2} + \left(B_{s_1} + B_{-s_1} - \frac{1}{C_{s_1}}\frac{dC_{s_1}}{dr}\right)\left(\frac{d\Psi_1^{s_1}}{dr} + B_{s_1}\,\Psi_1^{s_1}\right) + \left(\frac{dB_{s_1}}{dr} - B_{s_1}^2 - C_{s_1}C_{-s_1}\right)\Psi_1^{s_1} = 0\,, \tag{4.100}$$

which is the decoupled second order differential equation for $\Psi_1^{s_1}$. An analytical exact solution of the latter differential equation is, probably, unapproachable. Nevertheless, we can use (4.100) to infer the asymptotic forms of the solution near the infinity, $r \to \infty$, as well as near the horizon $r \to r_*$, where r_* is a root of the function f, namely $f(r_*) = 0$. In order to perform this analysis, we shall write the equation (4.100) as:

$$\frac{d^2\Psi_1^{s_1}}{dr^2} + h_1^{s_1}(r)\,\frac{d\Psi_1^{s_1}}{dr} + h_0^{s_1}(r)\,\Psi_1^{s_1} = 0\,, \tag{4.101}$$

where functions $h_1^{s_1}(r)$ and $h_0^{s_1}(r)$ can be obtained without great effort by comparing equations (4.100) and (4.102). Before proceeding, we should note that considering the solutions of the equation (4.102) in the limit $r \to \infty$ we must expand the function $h_1^{s_1}(r)$ up to order $r^{-(p+1)}$ and $h_0^{s_1}(r)$ up to order $r^{-(p+2)}$ if we want to know $\Psi_1^{s_1}$ up to order r^{-p}. This ensures that our research in this limit will be consistent. Once called attention to this important detail, we now can work out the asymptotic forms of the functions $h_1^{s_1}(r)$ and $h_0^{s_1}(r)$ in the region of interest.

Let us separate the analysis of these two cases. The motivation for this division is that the function $f(r)$ in the black hole line element (4.71) has a term that multiplies Λ becomes the dominant one as we approach the infinity, $r \to \infty$. Therefore, it is natural to guess that the cases of vanishing and non-vanishing Λ should be qualitatively different. Let us then consider first the case $\Lambda \neq 0$. Collecting the functions that that accompany $\frac{d^2\Psi_1^{s_1}}{dr^2}$ and $\Psi_1^{s_1}$ in equation (4.100) and then expanding them in powers of r^{-1}, we can find, after some algebra, the following asymptotic forms

$$\begin{aligned} h_0^{s_1}(r) &= \frac{(d-1)m^2}{\Lambda\,r^2} + \frac{i\,(d-1)\,s_1\,\omega}{\Lambda\,r^3} + O\left(\frac{1}{r^4}\right)\,, \\ h_1^{s_1}(r) &= \frac{1}{r} - \frac{s_1\,c_1}{m\,r^2} + \left[\frac{2\,(d-1)}{(d-3)\Lambda} - \frac{c_1^2}{m^2}\right]\frac{1}{r^3} + O\left(\frac{1}{r^4}\right)\,. \end{aligned} \tag{4.102}$$

Particularly, if we consider the expansion of $h_0^{s_1}(r)$ up to order r^{-2} and the expansion of $h_1^{s_1}(r)$ up to order r^{-1}, the integration of the equation (4.102) gives us the following asymptotic form:

$$\Psi_1^{s_1} \sim C_0 \sin\left[\frac{m\sqrt{d-1}}{\sqrt{\Lambda}}\log(r) + \psi_0\right], \qquad (4.103)$$

where C_0 and ψ_0 are arbitrary integration constants.

Now, let us consider the second case on which the cosmological constant is null, $\Lambda = 0$. In this case, when $d > 6$ one can see, after performing a subtle algebra, that the asymptotic forms of the functions $h_1^{s_1}(r)$ and $h_0^{s_1}(r)$are the following

$$\begin{aligned} h_0^{s_1}(r) &= \left[(d-3)^2\,\omega^2 - (d-3)\,m^2\right] \\ &+ \left[\frac{3}{4} + (d-3)\left(c_1^2 - \frac{i\,\omega\,c_1}{m}\right) - \frac{(d-3)^2 Q_m^2\, m^2}{4\,(d-5)} + \frac{(d-3)^3 Q_m^2 \omega^2}{2(d-5)}\right]\frac{1}{r^2} \\ &+ O\left(\frac{1}{r^3}\right), \\ h_1^{s_1}(r) &= -\frac{1}{r} - \frac{s_1\,c_1}{m\,r^2} + O\left(\frac{1}{r^3}\right). \end{aligned}$$

Note, however, that the case of vanishing cosmological constant, the well-known case d $= 4$ is qualitatively different from the higher-dimensional cases $d \geq 6$. Indeed, these formulas do not apply to the well-studied situation $d = 4$, in which case $h_0^{s_1}(r)$ has a term of order r^{-1} depending on M and Q_e and the function corresponding to the term of order r^{-2} has additional terms also depending on these constants. A completely analogous analysis can be done for $h_1^{s_1}(r)$. These considerations lead us to the following conclusion. For $\Lambda = 0$, the spinor field that represents a charged particle of spin 1/2 moving in the black hole (4.71) has qualitatively different fall off properties in the asymptotic infinity depending on whether $d = 4$ or $d \geq 6$.

Above, we describe the asymptotic behavior in the limit when the coordinate r assumes values near the infinity. However, a similar analysis can be done near the horizon. This is the case of the values of r for which the function f vanishes. In such case the coordinate r ceases to be reliable and we should change the radial coordinate to tortoise-like coordinates. In [74] these coordinates are useful in the study of the dynamical stability of the Kerr Metric. Besides such asymptotic behaviours, one can also look for approximate solutions valid in a broader domain by means of other approximation methods. In particular the Wentzel-Kramers-Brillouin approximation method (WKB) has been widely used in the pursuit of such solutions. In [95] the WKB method is used to obtain an approximate solution for the Dirac field on the Kerr spacetime, while on [99] it is utilized to solve the radial part of the Dirac equation in the Schwarzschild geometry. In both references, the WKB method is applied after transforming the radial second order differential equation into a Schrodinger equation. To know more, consult the references [97, 82].

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