Problem 1. Determine all complex numbers $\lambda$ for which there exist a positive integer $n$ and a real $n \times n$ matrix $A$ such that $A^2 = A^T$ and $\lambda$ is an eigenvalue of $A.$

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution. By taking squares,

$$A^4 = (A^2)^2 = (A^T)^2 = (A^2)^T = (A^T)^T = A,$$

so

$$A^4 - A = 0;$$

it follows that all eigenvalues of $A$ are roots of the polynomial $X^4 - X.$

The roots of $X^4 - X = X(X^3 - 1)$ are 0, 1 and $\frac{-1 \pm \sqrt{3}i}{2}$. In order to verify that these values are possible, consider the matrices

$$A_0 = (0), \quad A_1 = (1), \quad A_2 = \left(\begin{array}{c}
-\frac{1}{2} & \frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{array}\right), \quad A_4 = \left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{array}\right).$$

The numbers 0 and 1 are the eigenvalues of the $1 \times 1$ matrices $A_0$ and $A_1$, respectively. The numbers $\frac{-1 \pm \sqrt{3}i}{2}$ are the eigenvalues of $A_2$; it is easy to check that

$$A_2^2 = \left(\begin{array}{cc}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{array}\right) = A_2^T.$$

The matrix $A_4$ establishes all the four possible eigenvalues in a single matrix.

Remark. The matrix $A_2$ represents a rotation by $2\pi/3$.

Problem 2. Let $f: \mathbb{R} \to (0, \infty)$ be a differentiable function, and suppose that there exists a constant $L > 0$ such that

$$|f'(x) - f'(y)| \leq L|x - y|$$

for all $x, y$. Prove that

$$(f'(x))^2 < 2Lf(x)$$

holds for all $x.$

(Proposed by Jan Šustek, University of Ostrava)
**Solution.** Notice that $f'$ satisfies the Lipschitz-property, so $f'$ is continuous and therefore locally integrable.

Consider an arbitrary $x \in \mathbb{R}$ and let $d = f'(x)$. We need to prove $f(x) > \frac{d^2}{2L}$.

If $d = 0$ then the statement is trivial.

If $d > 0$ then the condition provides $f'(x-t) \geq d-Lt$; this estimate is positive for $0 \leq t < \frac{d}{L}$.

By integrating over that interval,

$$ f(x) > f(x) - f(x - \frac{d}{L}) = \int_0^{\frac{d}{L}} f'(x-t) \, dt \geq \int_0^{\frac{d}{L}} (d-Lt) \, dt = \frac{d^2}{2L}. $$

If $d < 0$ then apply $f'(x+t) \leq d + Lt = -|d| + Lt$ and repeat the same argument as

$$ f(x) > f(x) - f(x + |d|) = \int_0^{\frac{|d|}{L}} (-f'(x+t)) \, dt \geq \int_0^{\frac{|d|}{L}} (-|d| - Lt) \, dt = \frac{d^2}{2L}. $$

**Problem 3.** For any positive integer $m$, denote by $P(m)$ the product of positive divisors of $m$ (e.g. $P(6) = 36$). For every positive integer $n$ define the sequence

$$ a_1(n) = n, \quad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \ldots, 2016). $$

Determine whether for every set $S \subseteq \{1, 2, \ldots, 2017\}$, there exists a positive integer $n$ such that the following condition is satisfied:

For every $k$ with $1 \leq k \leq 2017$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.

(Proposed by Matko Ljulj, University of Zagreb)

**Solution.** We prove that the answer is yes; for every $S \subseteq \{1, 2, \ldots, 2017\}$ there exists a suitable $n$. Specially, $n$ can be a power of 2: $n = 2^w$ with some nonnegative integer $w_1$. Write $a_k(n) = 2^{w_k}$; then

$$ 2^{w_{k+1}} = a_{k+1}(n) = P(a_k(n)) = P(2^{w_k}) = 1 \cdot 2 \cdot 4 \cdots 2^{w_k} = 2^{w_k(w_k+1)}, $$

so

$$ w_{k+1} = \frac{w_k(w_k + 1)}{2}. $$

The proof will be completed if we prove that for each choice of $S$ there exists an initial value $w_1$ such that $w_k$ is even if and only if $k \in S$.

**Lemma.** Suppose that the sequences $(b_1, b_2, \ldots)$ and $(c_1, c_2, \ldots)$ satisfy $b_{k+1} = \frac{b_k(b_k+1)}{2}$ and $c_{k+1} = \frac{c_k(c_k+1)}{2}$ for $k \geq 1$, and $c_1 = b_1 + 2^m$. Then for each $k = 1, \ldots, m$ we have $c_k \equiv b_k + 2^{m-k+1}$ (mod $2^{m-k+2}$).

As an immediate corollary, we have $b_k \equiv c_k$ (mod 2) for $1 \leq k \leq m$ and $b_{m+1} \equiv c_{m+1} + 1$ (mod 2).

**Proof.** We prove the by induction. For $k = 1$ we have $c_1 = b_1 + 2^m$ so the statement holds. Suppose the statement is true for some $k < m$, then for $k+1$ we have

$$ c_{k+1} = \frac{c_k(c_k + 1)}{2} = \frac{(b_k + 2^{m-k+1})(b_k + 2^{m-k+1} + 1)}{2} $$

$$ = \frac{b_k^2 + 2^{m-k+2}b_k + 2^{2m-k+2} + b_k + 2^{m-k+1}}{2} $$

$$ = \frac{b_k(b_k + 1)}{2} + 2^{m-k} + 2^{m-k+1}b_k + 2^{2m-2k+2} = \frac{b_k(b_k + 1)}{2} + 2^{m-k} \quad \text{(mod $2^{m-k+1}$)}, $$

therefore $c_{k+1} \equiv b_{k+1} + 2^{m-(k+1)+1}$ (mod $2^{m-(k+1)+2}$).
Going back to the solution of the problem, for every $1 \leq m \leq 2017$ we construct inductively a sequence $(v_1, v_2, \ldots)$ such that $v_{k+1} = \frac{v_k(v_k+1)}{2}$, and for every $1 \leq k \leq m$, $v_k$ is even if and only if $k \in S$.

For $m = 1$ we can choose $v_1 = 0$ if $1 \in S$ or $v_1 = 1$ if $1 \notin S$. If we already have such a sequence $(v_1, v_2, \ldots)$ for a positive integer $m$, we can choose either the same sequence or choose $v'_1 = v_1 + 2^m$ and apply the same recurrence $v'_{k+1} = \frac{v'_k(v'_k+1)}{2}$. By the Lemma, we have $v_k \equiv v'_k \pmod{2}$ for $k \leq m$, but $v_{m+1}$ and $v_{m+1}$ have opposite parities; hence, either the sequence $(v_k)$ or the sequence $(v'_k)$ satisfies the condition for $m + 1$.

Repeating this process for $m = 1, 2, \ldots, 2017$, we obtain a suitable sequence $(w_k)$.

**Problem 4.** There are $n$ people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group $S$ of people such that at least $n/2017$ persons in $S$ have exactly two friends in $S$.

(Proposed by Rooholah Majdodin and Fedor Petrov, St. Petersburg State University)

**Solution.** Let $d = 1000$ and let $0 < p < 1$. Choose the set $S$ randomly such that each person is selected with probability $p$, independently from the others.

The probability that a certain person is selected for $S$ and knows exactly two members of $S$ is

$$q = \left(\frac{d}{2}\right) p^3(1-p)^{d-2}.$$ 

Choose $p = 3/(d + 1)$ (this is the value of $p$ for which $q$ is maximal); then

$$q = \left(\frac{d}{2}\right) \left(\frac{3}{d + 1}\right)^3 \left(\frac{d - 2}{d + 1}\right)^{d-2} =$$

$$= \frac{27d(d - 1)}{2(d + 1)^3} \left(1 + \frac{3}{d - 2}\right)^{-(d-2)} > \frac{27d(d - 1)}{2(d + 1)^3} \cdot e^{-3} > \frac{1}{2017}.$$ 

Hence, $E(|S|) = nq > \frac{n}{2017}$, so there is a choice for $S$ when $|S| > \frac{n}{2017}$.

**Problem 5.** Let $k$ and $n$ be positive integers with $n \geq k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_0$$

be a polynomial with complex coefficients such that

$$c_0c_{n-2} = c_1c_{n-3} = \ldots = c_{n-2}c_0 = 0.$$ 

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

(Proposed by Vsevolod Lev and Fedor Petrov, St. Petersburg State University)

**Solution.** Let $M = \{z : z^n = 1\}$, $A = \{z \in M : f(z) \neq 0\}$ and $A^{-1} = \{z^{-1} : z \in A\}$. We have to prove $|A| \geq k$.

**Claim.**

$$A \cdot A^{-1} = M.$$

That is, for any $\eta \in M$, there exist some elements $a, b \in A$ such that $ab^{-1} = \eta$.

**Proof.** As is well-known, for every integer $m$,

$$\sum_{z \in M} z^m = \begin{cases} n & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$$
Define $c_{n-1} = 1$ and consider
\[
\sum_{z \in M} z^2 f(z) f(\eta z) = \sum_{z \in M} z^2 \sum_{j=0}^{n-1} c_j z^j \sum_{\ell=0}^{n-1} c_\ell (\eta z)^\ell = \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} c_j c_\ell \eta^\ell \sum_{z \in M} z^{j+\ell+2} =
\]
\[
= \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} c_j c_\ell \eta^\ell \sum_{z \in M} \begin{cases} 
 0 & \text{if } n \n j + \ell + 2 
 1 & \text{otherwise}
\end{cases} = c_{n-1}^2 n + \sum_{j=0}^{n-2} c_j c_{n-2-j} \eta^{n-2-j} n = n \neq 0.
\]

Therefore there exists some $b \in M$ such that $f(b) \neq 0$ and $f(\eta b) \neq 0$, i.e. $b \in A$, and $a = \eta b \in A$, satisfying $ab^{-1} = \eta$.

By double-counting the elements of $M$, from the Claim we conclude
\[
|A|(|A| - 1) \geq |M \setminus \{1\}| = n - 1 \geq k^2 - 3k + 3 > (k - 1)(k - 2)
\]
which shows $|A| > k - 1$. 