Physical measures for certain partially hyperbolic attractors on 3-manifolds

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Introduction

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It is interesting to point results for generic partially hyperbolic attractors, including cases where the central Lyapunov exponent is zero.

Introduction

Definition

- Given a f-invariant measure μ, the **basin of** μ is the set $B(\mu) = \left\{ x \in M |\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} = \mu \right\}$.
- μ is a **physical measure** if $\operatorname{vol}\left(B(\mu)\right) > 0$.
- Λ is an **attractor** if there exists an open set $U \subset M$ containing Λ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n\geq 0} f^n(U)$.
- Λ is **partially hyperbolic** if there exists constants $\lambda_{ss}^+ < \lambda_c^- < \lambda_c^+ < \lambda_{uu}^-$, $C > 1$ and a Df-invariant splitting $T_xM = E_x^{ss} \oplus E_x^c \oplus E_x^{uu}$ for every $x \in \Lambda$ such that:

$$
\begin{array}{rcll}
 & ||Df^n v|| & < C(\lambda_{ss}^+)^n ||v|| & v \in E_x^{ss} - \{0\} \\
C^{-1}(\lambda_c^-)^n ||v|| < & ||Df^n v|| & < C(\lambda_c^+)^n ||v|| & v \in E_x^{c} - \{0\} \\
C^{-1}(\lambda_{uu}^-)^n ||v|| < & ||Df^n v|| & & v \in E_x^{uu} - \{0\}
\end{array}
$$

Main Result

Theorem A

Consider $f : M \to M$ diffeomorphism of class C^r, $r \geq 2$, $\dim M = 3$ and Λ_0 partially hyperbolic attractor. Suppose that Λ_0 is dynamically coherent and:

(H1) Transversality between unstable leaves via stable holonomy;

(H2) Central direction neutral;

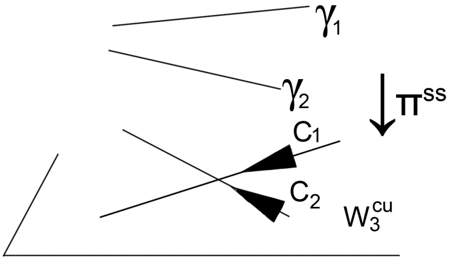
(H3') The stable holonomy h^{ss} is of class Lipschitz.

Then f admits a finite number of physical measures supported in Λ, whose union of their basins has full measure in $B(\Lambda)$.

Definitions and Statements

Definition

We say that holds the **Transversality Condition** if there exist a constant ϵ_0 and functions $\theta, r : (0, \epsilon_0) \to \mathbb{R}^+$ such that: For every unstable curves γ_1, γ_2 and center-unstable manifold W_3^{cu} with $d^{ss}(\gamma_i, W_3^{cu}) < \epsilon_0$, $i = 1, 2$, and $d^{ss}(\gamma_1, \gamma_2) > \epsilon$, the curves $\pi^{ss}(\gamma_1)$ and $\pi^{ss}(\gamma_2)$ are $\theta(\epsilon)$-transversal in neighborhoods of radius $r(\epsilon)$.



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Considering the constants λ_c^- , λ_c^+ , λ_{uu}^- of the definition of partial hyperbolicity, we say that Λ has **central direction neutral** if $\lambda_c^- < 1 < \lambda_c^+$ and

$$\frac{\lambda_c^+}{(\lambda_c^-)^2 \cdot \lambda_{uu}^-} < 1$$

This condition implies that $df|_{E^{cu}}$ is area expanding.
It includes situations where the central Lyapunov exponent is null and, therefore, we can't use Pesin's theory.

Main Results

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Corollary B

If f satisfies the conditions of Theorem A and:

(H3) $f \to \mathcal{F}_f^{ss}$ is continuous in the C^1-topology in f_0.

Then the same conclusion is valid for every f in a neighbourhood $\mathcal{U}$ of f_0.

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Theorem C

There exists $f_0 : M^3 \to M^3$ and a partially hyperbolic attractor Λ_0 that is robustly nonhyperbolic, robustly dynamically coherent and satisfies the hypothesis (H1), (H2) and (H3).

Tools

Definition (of $||\cdot||_r$)

Given a finite measure ν defined in a surface W and $r > 0$, define the **semi-norm:** $||\nu||^2_{W,r} = \frac{1}{r^4}\int_W \nu(B(z,r))^2 dm_W(z)$

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Check that $\pi^{ss}_*\mu(B^{cu}(x,r)) \leq Kr^2$ when $r \to 0^+$ for a.e. x, we obtain that $\pi^{ss}_*\mu << m^{cu}$ and, for every ergodic u-Gibbs μ this absolute continuity implies that μ is a physical measure.

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Lemma (Criteria of absolute continuity)

If $\liminf_{r\to 0^+} ||\nu||_{W^{cu},r} \leq L$, then ν is absolutely continuous with respect to m^{cu} and $\left|\left|\frac{d\nu}{dm^{cu}}\right|\right|_{L^2} \leq C_L$

Tools

Consider a finite number of boxes $(C_i, W_i, \tilde{W}_i, \pi_i^{ss})$, $\pi_i^{ss} : C_i \to W_i$, such that Λ is covered by $\{\pi_i^{-1}(W_i)\}_i$, and let $\mathcal{W}$ be $\{W_1, \cdots, W_{s_0}\}$.

Definition (of $|||\cdot|||_r$)

Define $|||\nu|||_r$ by:

$$|||\mu|||_{\mathcal{W},r} := \max_{i=1,\cdots,s_0} \{||(\pi_i^{ss})_* \mu||_{W_i,r}\}$$

The Main Inequality

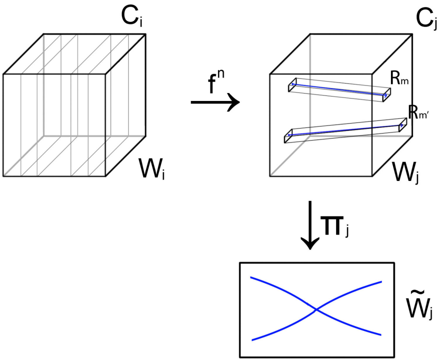
Proposition (Main Inequality)

There exist constants $B > 0$ and $\sigma > 1$ such that for every $n \in \mathbb{N}$, there exists $D_n > 0$, $r_n > 0$ and $c_n > 1$ such that for every ergodic u-Gibbs μ and $r \leq r_n$, we have:

$$|||f_*^n \mu|||_r^2 \leq \frac{B}{\sigma^n} |||\mu|||_{c_n r}^2 + D_n |\mu|^2$$

$$\begin{array}{lcl} |||\cdot||| & \rightarrow & \text{regularity (similar to } L^2\text{-norm)} \\ |\cdot| & \rightarrow & \text{size (similar to } L^1\text{-norm)} \end{array}$$

The cancelation mechanism:



Transversality implies: $\langle \nu_i, \nu_j \rangle_{W_j,r} < C_\theta \cdot |\nu_i| \cdot |\nu_j|$ if the measures ν_i and ν_j are absolutely continuous with respect to length on θ-transversal curves.

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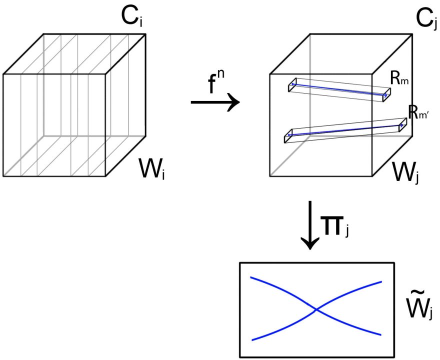

$$m \neq m' \Rightarrow \langle (\pi_j f_m)_* \mu, (\pi_j f_{m'})_* \mu \rangle_{W_j,r} < K_n \cdot |(\pi_j)_*(f_m)_* \mu| \cdot |(\pi_j)_*(f_{m'})_* \mu|$$

Proof of Theorem A

As a consequence of the Main Inequality, we have:

Proposition

Every ergodic u-Gibbs μ projects into an absolutely continuous measure in W_i by π_i and $\left\|\frac{d((\pi_i)_*\mu)}{dm^{cu}}\right\|_{L^2} \leq K$.

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So μ is a physical measure. An uniform lower bound on the Lebesgue measure of the basin implies that there are at most finite physical measures. Moreover, the union of the basins of the physical measures has full Lebesgue measure.

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Suppose that $f \to \mathcal{F}_f^{ss}$ is continuous in the C^1-topology, there exists some $l > 0$ and $\epsilon_1 > 0$ such that: for every pair of unstable curves γ_1, γ_2 with $d^{ss}(\gamma_1, \gamma_2) \in [a, la]$ for some $a > 0$ and every center-unstable manifold W_3^{cu} with $d^{ss}(\gamma_i, W_3^{cu}) \leq \epsilon_1$, the curves $\pi^{ss}\gamma_1$ and $\pi^{ss}\gamma_2$ are θ-transversal in neighborhood of radius r. Then it is valid the Hypothesis of Transversality.

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Proposition above states that it is enough to guarantee the transversality in a certain "scale". The openess follows by the *continuity of* $f \to d\pi^{ss}_f E^{uu}_f(x)$.

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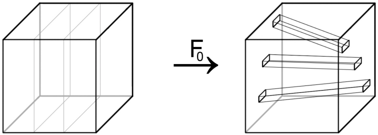

This attractor is constructed deforming a hyperbolic solenoid.

- We consider a 1-parameter family $F_{\mu,n}$ deforming F_0^n in the central direction changing the index of a fixed point, $\mu \in [0,1]$, $n \in \mathbb{N}$.

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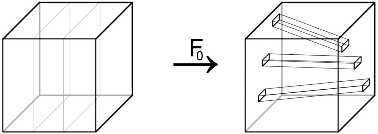

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Proposition 1

There exists $n_1 \in \mathbb{N}$ such that the Transversality Condition is valid for $F_{\mu,n}$, for every $\mu \in [0,1]$ and every $n \geq n_1$.

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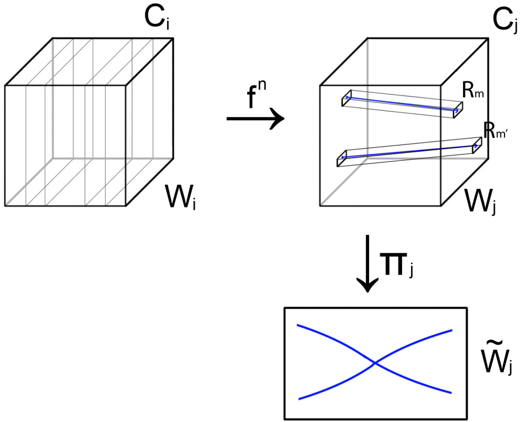
- Consider α-**Holder stable foliations**.

Thank you!


Cᵢ
fⁿ
Cⱼ
Rₘ
Rₘ′
Wᵢ
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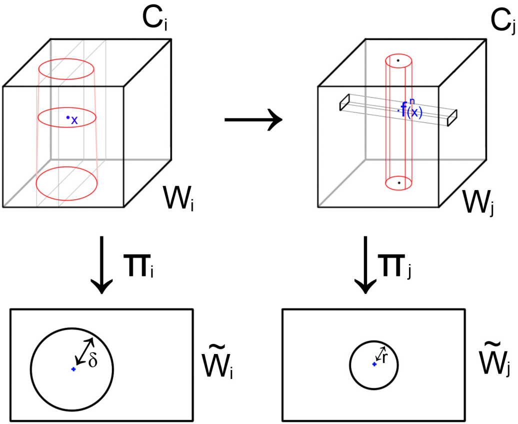
πⱼ
W̃ⱼ


C_i
x
W_i
C_j
$f^n(x)$
W_j
π_i
π_j
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$\tilde{W}_i$
r
$\tilde{W}_j$